Area A – CSE Prelim Exam
CSE 386M / CSE 386L

May 31, 2018

Solve the following six problems in 3 hours.

1. A linear algebra problem. Let $\boldsymbol{\omega} = (1, 1, 1) \in \mathbb{R}^3$. Consider the map:

$$\mathbb{R}^3 \ni \boldsymbol{x} \to \boldsymbol{y} = A\boldsymbol{x} := \boldsymbol{\omega} \times \boldsymbol{x} \in \mathbb{R}^3 \,.$$

- Prove that map A is linear.
- Find the matrix representation of map A in the canonical basis $\boldsymbol{e}_i$.
- Determine inverse map A^{-1}.
- Determine transpose map A^T and its matrix representation with respect to (wrt) the dual basis $\boldsymbol{e}_j^*$.
- Determine adjoint map A^* with respect to canonical inner product in $\mathbb{R}^n$.
- Consider a *weighted inner product* in $\mathbb{R}^3$:

$$(\boldsymbol{x}, \boldsymbol{y})_w := x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 \,,$$

  and determine the adjoint of A wrt to the weighted inner product.

Solution:

- This follows immediately from the fact that the cross product is a bilinear operation. It also follows from its explicit representation:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,.$$

- See above.
- It does not exist. A has a non-trivial null space,

$$\mathcal{N}(A) = \{\alpha\boldsymbol{\omega} = (\alpha, \alpha, \alpha) \,:\, \alpha \in \mathbb{R}\} \,.$$

- Matrix representation of A^T in the canonical dual basis equals simply the transpose of matrix representation of A,

$$A^T == \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = -A$$

- Just the transpose, $A^* = A^T = -A$. A is so-called *skew-adjoint*.
- This is the only part where you have to do some work.

$$\begin{aligned}(A\boldsymbol{x}, \boldsymbol{y})_w &= (x_3 - x_2)y_1 + 2(x_1 - x_3)y_2 + 3(x_2 - x_1)y_3 \\ &= x_1(2y_2 - 3y_3) + x_2(-y_1 + 3y_3) + x_3(y_1 - 2y_2) \\ &= x_1(2y_2 - 3y_3) + 2x_2(-\tfrac{1}{2}y_1 + \tfrac{3}{2}y_3) + 3x_3(\tfrac{1}{3}y_1 - \tfrac{2}{3}y_2) \\ &= (\boldsymbol{x}, A^*\boldsymbol{y})_w\end{aligned}$$

so the matrix representation of the new adjoint in the canonical basis is:

$$\begin{pmatrix} 0 & 2 & -3 \\ -\frac{1}{2} & 0 & \frac{3}{2} \\ \frac{1}{3} & -\frac{2}{3} & 0 \end{pmatrix}.$$

2. An integration exercise.

   - State the Lebesgue Dominated Convergence Theorem.
   - Let $\Omega \subset \mathbb{R}^n$ be an arbitrary *unbounded* open set, and let $f \in L^1(\Omega)$. Prove that

$$\int_{\Omega - B(0,n)} f(x)\,dx \to 0 \quad \text{as } n \to \infty\,.$$

   where $B(0,n)$ denotes the ball centered at 0 with radius n.

Solution:

- See the book.
- Consider

$$f_n(x) := \begin{cases} 0 & x \in \Omega \cap \bar{B}(0,n) \\ f(x) & x \in \Omega - B(0,n) \end{cases}$$

  Obviously, $f_n(x) \to 0$ as $n \to \infty$, and $|f_n(x)| \leq |f(x)|$, so $|f(x)|$ provides a dominating function. By the Lebesque Theorem,

$$\int_{\Omega - B(0,n)} f(x)\,dx = \int_{\Omega} f_n(x)\,dx \to 0\,.$$

3. Application of Banach Contractive Map Theorem.

- State the Contractive Map Theorem. Make sure to list *all* assumptions.
- Consider the following initial value problem:

$$\begin{cases} \frac{dx}{dt} = \frac{1}{3}x^{-2}(t-1)^{-1}, & t \in (0,T) \\ x(0) = 1 \end{cases}$$

where, at this point, T is unknown. Use elementary means to solve the problem. Comment on a maximum T for which the solution exists in interval $(0,T)$.

- Use the Contractive Map Theorem to prove that the solution exists (kind of after dinner exercise for this example) and provide a concrete estimate for interval length T.

Solution:

- See the book.
- Use separation of variables to obtain:

$$3x^2 dx = \frac{dt}{x-1}$$

to obtain:

$$x^3|_1^{x(t)} = \ln|t-1||_0^t ,$$

and, finally,

$$x(t) = [\ln|t-1| + 1]^{\frac{1}{3}} .$$

Verify that the solution indeed satisfies the ODE and IC. The solution has a blow up at $t = 1$.

- The IVP is equivalent to the integral equation:

$$x(t) = 1 + \frac{1}{3}\int_0^t x^{-2}(s)\,(s-1)^{-1}\,ds =: (Ax)(t)$$

and $x(t)$ is a solution to the integral eqn iff it is a fixed point of operator A. More precisely, we will identify a set $D \subset C([0,T])$ (with T to be determined !) such that a) A is well defined, i.e. it maps D into itself, and b) A is a contraction on D.

The flux $f(x,s) = \frac{1}{3}x^{-2}(t-1)^{-1}$ is undefined for $x = 0$ and $t = 1$. This motivates use to seek $T < 1$ and define,

$$D := \{x \in C([0,T]) \,:\, x(t) \geq \frac{1}{2}\} .$$

The $\frac{1}{2}$ in the definition above is somehow arbitrary, could have used any positive constant. Definition of D is consistent with the IC and all functions from D are bounded pointwise by 2. Condition a) requires that

$$1 + \frac{1}{3}\int_0^t x^{-2}(s)(s-1)^{-1}\,ds \geq \frac{1}{2} \quad \text{for } t < T .$$

Equivalently,

$$|\int_0^t x^{-2}(s)(s-1)^{-1}\,ds| \le \frac{3}{2}$$

Working on the sufficient side, we get

$$|\int_0^t x^{-2}(s)(s-1)^{-1}\,ds| \le \int_0^t \underbrace{|x^{-2}(s)|}_{\le 4}|(s-1)^{-1}|\,ds \le 4\int_0^t \frac{ds}{1-s} = -4\ln(1-t) \le \frac{3}{2}.$$

or, equivalently,

$$-\ln(1-t) \le \frac{3}{8}.$$

Condition b) requires that

$$|x_1(t)-x_2(t)| \le \int_0^t |x_1^{-2}(s)-x_2^{-2}(s)|(1-s)^{-1}\,ds \le C\|x_1-x_2\|_{C([0,T])}$$

with $C<1$. By the Mean-Value Theorem, for $f(x)=x^{-2}$,

$$f(x_2)-f(x_1) = f'(\xi)(x_2-x_1) \quad \text{where } \xi \in (x_1,x_2)\,,$$

with $f'(x) = -2x^{-3}$ and $|-2x^{-3}| \le 16$ for $x \in D$. Consequently, the integral above is estimated by,

$$\frac{16}{3}\underbrace{\int_0^t \frac{ds}{1-s}}_{=-\ln(1-t)}\underbrace{\max_{t\in[0,T]}|x_1(t)-x_2(t)|}_{=\|x_1-x_2\|_{C([0,T])}}$$

so the contraction condition is satisfied if

$$-\ln(1-t) < \frac{3}{16}.$$

Of the two conditions, the second one is more restrictive and it leads to the final condition for T,

$$T < 1-e^{-\frac{3}{16}}.$$

4. Let u and w be two scalar fields in $\mathbb{R}^2$.

   - Consider a rectangular domain $\Omega = (a_1, a_2) \times (b_1, b_2) \subset \mathbb{R}^2$, i.e, if $(x, y) \in \Omega$ then $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$. First **prove** that

$$\int_\Omega \frac{\partial u}{\partial y} w \, d\Omega = -\int_\Omega u \frac{\partial w}{\partial y} \, d\Omega + \int_{\partial\Omega} u \, w \, n_y \, ds,$$

     where n_y is the y-component of the unit outward normal vector $\mathbf{n}$ on the boundary $\partial\Omega$.

   - Using the above (and/or similar) result to **prove** the following First Green Identity

$$\int_\Omega \nabla \cdot \mathbf{F} w \, d\Omega = -\int_\Omega \mathbf{F} \cdot \nabla w \, d\Omega + \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, w \, ds,$$

     where $\mathbf{F}$ is a vector field in $\mathbb{R}^2$.

   - **Derive** the Gauss divergence theorem

$$\int_\Omega \nabla \cdot \mathbf{F} \, d\Omega = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, ds,$$

Solution:

- We have

$$\int_\Omega \frac{\partial u}{\partial y} w \, d\Omega = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} \frac{\partial u}{\partial y} w \, dy \right) \, dx$$
$$= \int_{a_1}^{a_2} \left(-\int_{b_1}^{b_2} u \frac{\partial w}{\partial y} \, dy + u\,(x, b_2)\, w\,(x, b_2) - u\,(x, b_1)\, w\,(x, b_1) \right) \, dx$$
$$= -\int_\Omega u \frac{\partial w}{\partial y} \, d\Omega + \int_{\partial\Omega} u \, w \, n_y \, ds,$$

  where we have used the fact that $n_y = -1, 0, 1$ on the lower y-boundary, x-boundaries, and upper y-boundary, respectively. Here, we also use the fact that $dx = -n_y ds$ on these boundaries.

- Easy:

$$\int_\Omega \frac{\partial F_1}{\partial x} w \, d\Omega = -\int_\Omega F_1 \frac{\partial w}{\partial x} \, d\Omega + \int_{\partial\Omega} F_1 \, w \, n_x \, ds,$$

$$\int_\Omega \frac{\partial F_2}{\partial y} w \, d\Omega = -\int_\Omega F_2 \frac{\partial w}{\partial y} \, d\Omega + \int_{\partial\Omega} F_2 \, w \, n_y \, ds,$$

- take $w = const$

5. Consider a flexible string with mass density (mass per unit length) $\rho(x)$ tied between $x = 0$ and $x = \ell$. The string is assumed to be under a constant tension τ at any point along the string at any time. It can be shown that the potential energy and the kinetic energy of the string are given by

$$V = \frac{\tau}{2}\int_0^\ell y_x^2\, dx, \quad \text{and } T = \frac{1}{2}\int_0^\ell \rho\left(x\right) y_t^2\, dx,$$

where y is the vertical displacement of the string, y_x and y_t are partial derivative of y with respect to x and t.

- According the Hamilton's principle, the equation of motion of the string is given by the Euler-Lagrange equation of the following functional

$$\int_0^T \left(T - V\right)\, dt.$$

Derive in details the equation of motion for the string.

- Now assume the density ρ is constant. **Solve in details** for the displacement of the string given the initial displacement

$$y_0\left(x\right) = sin\left(\frac{2\pi}{\ell}x\right),$$

and zero initial velocity $y_t\left(x\right) = 0$.

Solution:

- For

$$I = \int_t\int_x f\left(x, t, y, y_x, y_t\right)\, dx\, dt,$$

the Euler-Lagrange equation for fixed end points of y is given as

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial y_t}\right) = 0.$$

Applying this result for our case yields

$$\tau y_{xx} - \rho y_{tt} = 0,$$

which is the wave equation.

- Solve the wave equation using separation of variables

$$y = v(x)w(t)$$

to get to

$$\frac{v''}{v} = \frac{1}{\alpha^2}\frac{w''}{w} = -k^2,$$

where $\alpha^2 = \frac{\tau}{\rho}$. We then have

$$v_n = \sin\left(\frac{n\pi}{\ell}x\right), \quad w_n = \cos\left(\frac{\alpha n\pi}{\ell}t\right),$$

and the general solution is then

$$y(x,t) = \sum_n e_n \sin\left(\frac{n\pi}{\ell}x\right)\cos\left(\frac{\alpha n\pi}{\ell}t\right).$$

For

$$y_0\,(x) = sin\left(\frac{2\pi}{\ell}x\right),$$

only one term survives and hence the solution is

$$y(x,t) = sin\left(\frac{2\pi}{\ell}x\right)\cos\left(\frac{\alpha 2\pi}{\ell}t\right).$$

6. Let $\Omega = (0,1) \times (0,1)$.

- **Solve in details** the following eigenvalue problem

$$-\Delta w = \lambda w \text{ in } \Omega$$

with the homogeneous boundary condition $w(x,y) = 0$ on $\partial\Omega$. Moreover, **show** that the operator $-\Delta$ with homogeneous boundary condition is a self-adjoint operator. As a result, **argue in details** that "any function" $v(x,y)$ can be expressed as

$$v(x,y) = \sum_{m=1,n=1}^{\infty,\infty} v_{mn}\phi_{mn}(x,y),$$

where $\phi_{mn}(x,y)$ are eigenfunctions of $-\Delta$ and v_{mn} are coefficients in the expansion.

- Denote $(\phi_{mn}(x,y), \lambda_{mn})$ as eigenpairs of the previous question. Consider the following PDE

$$-\Delta w = f(x,y) \text{ in } \Omega,$$

and boundary condition $w(x,y) = 0$ on $\partial\Omega$. **Solve in details** for the solution of this equation for a general $f(x,y)$ and then **deduce** the solution for $f(x,y) = 2sin(\pi x)\sin(2\pi y)$.

HINT: You may want to use the following

$$f(x,y) = \sum_{m=1,n=1}^{\infty,\infty} f_{mn}\phi_{mn}(x,y),$$

and

$$-\Delta v(x,y) = -\sum_{m=1,n=1}^{\infty,\infty} v_{mn}\Delta\phi_{mn}(x,y) = \sum_{m=1,n=1}^{\infty,\infty} v_{mn}\lambda_{mn}\phi_{mn}(x,y).$$

Solution:

- By separation of variables $w = X(x)Y(x)$, we conclude that

$$\lambda_{mn} = \pi^2\left(m^2 + n^2\right), \quad \phi_{mn} = \sin(m\pi x)\sin(n\pi y).$$

The self-adjointness is clear by integration by parts two times

- Using the hint and the orthogonality of the eigenfunctions, we conclude that

$$w_{mn} = \frac{f_{mn}}{\lambda_{mn}},$$

and for $f(x,y) = 2sin(\pi x)\sin(2\pi y)$, we see that

$$w = \frac{f_{12}}{\lambda_{12}} sin(\pi x)\sin(2\pi y) = \frac{2}{5\pi^2} sin(\pi x)\sin(2\pi y)$$

CSEM Preliminary Exam - Area A-CSE
CSE 386M / CSE 386L
May 30, 2019

Solve the following six problems in 3 hours.

Part I - CSE 386M

1. A linear algebra problem. Let $\boldsymbol{a} = (1,1,1) \in \mathbb{R}^3$. Consider the map:

$$\mathbb{R}^3 \ni \boldsymbol{x} \to \boldsymbol{y} = A\boldsymbol{x} := \boldsymbol{a}(\boldsymbol{a} \cdot \boldsymbol{x}) \in \mathbb{R}^3$$

where the "dot" denotes the canonical inner product in $\mathbb{R}^3$.

- Prove that map A is linear.
- Find the matrix representation of map A in the canonical basis $\boldsymbol{e}_i$.
- Determine rank and the null space of A.
- Determine transpose map A^T and its matrix representation with respect to (wrt) the dual basis $\boldsymbol{e}_j^*$.
- Determine adjoint map A^* with respect to canonical inner product in $\mathbb{R}^n$.
- Consider a *weighted inner product* in $\mathbb{R}^3$:

$$(\boldsymbol{x}, \boldsymbol{y})_w := x_1 y_1 + 2x_2 y_2 + 3x_3 y_3 \,,$$

and determine the adjoint of A wrt to the weighted inner product.

Solution:

- This follows immediately from the fact that the dot product is a bilinear operation.
- We have

$$A\boldsymbol{e}_i = \boldsymbol{a} \quad \text{for } i = 1, 2, 3.$$

Consequently, the matrix representation of operator A is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

- The range of A is the line spanned by vector $\boldsymbol{a}$. Consequently, rank $A = 1$. The null space consist of all vectors orthogonal to $\boldsymbol{a}$,

$$\mathcal{N}(A) = \{\boldsymbol{x} \in \mathbb{R}^3 \,:\, \boldsymbol{x} \cdot \boldsymbol{a} = 0\}.$$

- Matrix representation of A^T in the canonical dual basis equals simply the transpose of matrix representation of A,
$$A^T = A\,.$$
- Just the transpose, $A^* = A^T = A$. A is *self-adjoint*.
- This is the only part where you have to do some work.
$$\begin{aligned}(A\boldsymbol{x},\boldsymbol{y})_w &= (x_1+x_2+x_3)y_1 + 2(x_1+x_2+x_3)y_2 + 3(x_1+x_2+x_3)y_3\\ &= x_1(y_1+2y_2+3y_3) + x_2(y_1+2y_2+3y_3) + x_3(y_1+2y_2+3y_3)\\ &= x_1(y_1+2y_2+3y_3) + 2x_2\frac{1}{2}(y_1+2y_2+3y_3) + 3x_3\frac{1}{3}(y_1+2y_2+3y_3)\\ &= (\boldsymbol{x}, A^*\boldsymbol{y})_w\end{aligned}$$
so the matrix representation of the new adjoint in the canonical basis is:
$$\begin{pmatrix} 1 & 2 & 3\\ \frac{1}{2} & 1 & \frac{3}{2}\\ \frac{1}{3} & \frac{2}{3} & 1\end{pmatrix}.$$

2. A topology exercise.

   - Define the notion of equivalent norms and explain why any two equivalent norms generate the same (norm) topology.
   - Recall that any two norms defined on a finite-dimensional space are equivalent, consider two norms on $\mathbb{R}^N$:
$$\|\boldsymbol{x}\|_1 := \sum_{i=1}^N |x_i| \qquad \|\boldsymbol{x}\|_p := (\sum_{i=1}^N |x_i|^p)^{1/p}, \quad p > 1\,,$$
and estimate the corresponding equivalence constants.

Solution:

- See the book.
- We need to find constants C_1 and C_2 such that
$$\|\boldsymbol{x}\|_p \le C_1\|\boldsymbol{x}\|_1 \qquad \text{and} \qquad \|\boldsymbol{x}\|_1 \le C_2\|\boldsymbol{x}\|_p\,.$$
Let us start with the first one. As both norms are homogenous[1], it is sufficient to prove the inequality for $\boldsymbol{x}$ that are unit in norm $\|\cdot\|_1$, i.e.
$$\sum_{i=1}^N |x_i| = 1 \qquad \Rightarrow \qquad |x_i| \le 1, \quad i = 1, \ldots, N\,.$$
This implies that ($p > 1$),
$$|x_i|^p \le |x_i| \qquad \Rightarrow \qquad \sum_{i=1}^N |x_i|^p \le \sum_{i=1}^N |x_i| = 1$$
and, consequently,
$$(\sum_{i=1}^N |x_i|^p)^{1/p} \le 1 = \sum_{i=1}^N |x_i|\,,$$
so $C_1 \le 1$. We proceed similarly with the second inequality, normalizing both sides this time with $\|\boldsymbol{x}\|_p$. We have,
$$\|\boldsymbol{x}\|_p = (\textstyle\sum_{i=1}^N |x_i|^p)^{1/p} = 1 \qquad \Rightarrow \qquad \textstyle\sum_{i=1}^N |x_i|^p = 1 \qquad \Rightarrow \qquad |x_i|^p \le 1, \quad i = 1, \ldots, N$$
$$\Rightarrow \qquad |x_i| \le 1, \quad i = 1, \ldots, N\,.$$
This implies that
$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^N |x_i| \le N = N\|\boldsymbol{x}\|_p$$
so we have the estimate:
$$C_2 \le N\,.$$

[1] We can divide both sides by $\|\boldsymbol{x}\|_1$.

3. Application of Banach Contractive Map Theorem.

- State the Contractive Map Theorem. Make sure to list *all* assumptions.
- Consider the following initial value problem:

$$\begin{cases} \frac{dx}{dt} = \frac{t}{x+1}, & t \in (0,T) \\ x(0) = 1 \end{cases}$$

where, at this point, T is unknown. Use elementary means to solve the problem. Comment on a maximum T for which the solution exists in interval $(0,T)$.

- Use the Contractive Map Theorem to prove that the solution exists (kind of after dinner exercise for this example) and provide a concrete estimate for interval length T.

Solution:

- See the book.
- Use separation of variables to obtain:

$$(x+1)dx = tdt$$

which leads to

$$\frac{(x+1)^2}{2}|_1^x = \frac{t^2}{2}|_0^t$$

and, finally,

$$x(t) = -1 \pm \sqrt{4+t^2}\,.$$

Solution with the minus sign does not satisfy the IC so it must be rejected. We obtain,

$$x(t) = -1 + \sqrt{4+t^2}\,.$$

Verify that the solution indeed satisfies the ODE and IC. The solution exists for any $T > 0$.

- The IVP is equivalent to the integral equation:

$$x(t) = 1 + \int_0^t \frac{s}{x(s)+1}\,ds =: (Ax)(t)$$

and $x(t)$ is a solution to the integral eqn iff it is a fixed point of operator A. More precisely, we will identify a set $D \subset C([0,T])$ (with T to be determined !) such that a) A is well defined, i.e. it maps D into itself, and b) A is a contraction on D.
The flux $f(x,s) = \frac{s}{x+1}$ is undefined for $x = -1$. This motivates to define,

$$D := \{x \in C([0,T]) \,:\, x(t) \geq 0\}\,.$$

Definition of D is consistent with the IC. Condition a) is now automatically satisfied for any T. Indeed,

$$x \in D \quad \Leftrightarrow \quad x(s) \geq 0 \quad \Rightarrow \quad \frac{s}{x(s)+1} \geq 0 \quad \Rightarrow \quad (Ax)(t) \geq 0 \quad \forall t \quad \Rightarrow \quad Ax \in D\,.$$

Condition b) requires that

$$|(Ax_1)(t) - (Ax_2)(t)| = |\int_0^t \left(\frac{s}{x_1(s)+1} - \frac{s}{x_2(s)+1} \right) ds| \overset{?}{\leq} C \underbrace{\max_{t \in [0,T]} |x_1(t) - x_2(t)|}_{=:\|x_1 - x_2\|},$$

with $C < 1$. For $x_1, x_2 \in D$, we have,

$$|\frac{s}{x_1+1} - \frac{s}{x_2+1}| = |\frac{s(x_2 - x_1)}{(x_1+1)(x_2+1)}| \leq s\|x_1 - x_2\|\,.$$

Consequently,

$$|\int_0^t \left(\frac{s}{x_1(s)+1} - \frac{s}{x_2(s)+1} \right) ds| \leq \int_0^t s\,ds\,\|x_1 - x_2\| = \frac{t^2}{2}\,\|x_1 - x_2\|\,.$$

For

$$\frac{T^2}{2} < 1 \quad \Leftrightarrow \quad T < \sqrt{2}$$

operator A is a contraction.

Part II - CSE 386L

4. Figure 1 presents a mapping between a reference triangle ($\boldsymbol{I}$) in $\boldsymbol{\xi} = (r, s)$-coordinates (computational) and an arbitrary triangle ($\boldsymbol{D}$) described by three given vertices $\boldsymbol{v}^1 = \left(x^1, y^1\right), \boldsymbol{v}^2 = \left(x^2, y^2\right), \boldsymbol{v}^3 = \left(x^3, y^3\right)$ in the $\boldsymbol{x} = (x, y)$-coordinates.

   This mapping is pervasive in numerical methods for partial differential equations, especially the finite element methods. For this case, the map from computational domain $\boldsymbol{I}$ to the physical domain $\boldsymbol{D}$ is given as

$$\boldsymbol{x} = (x, y) = \boldsymbol{\Psi}(r, s) = -\frac{r+s}{2}\boldsymbol{v}^1 + \frac{r+1}{2}\boldsymbol{v}^2 + \frac{s+1}{2}\boldsymbol{v}^3$$

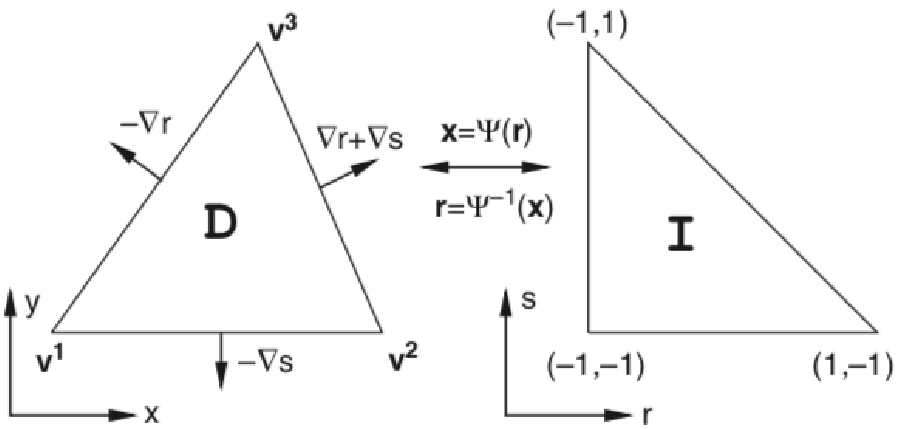


Figure 1: Mapping between computational triangle in the (r, s)-coordinates and the physical triangle in the (x, y) coordinates.

- Show that $\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta_{ij}$, where $\boldsymbol{g}_j, j = 1, 2$ are covariant vectors and $\boldsymbol{g}^i, i = 1, 2$ are contravariant vectors
- Express the covariant and contravariant vectors in terms of $\boldsymbol{v}^k$, $k = 1, 2, 3$.
- Let $\boldsymbol{v}^1 = (0, 0), \boldsymbol{v}^2 = (1, 1), \boldsymbol{v}^3 = (0, 1)$. Which point (x^*, y^*) in $\boldsymbol{D}$ is $(r, s) = (0, 1/2) \in \boldsymbol{I}$ mapped to? Compute the covariant and contravariant vectors at (x^*, y^*) and plot these vectors in $\boldsymbol{D}$.

Solution:

- See in lecture notes, in particular:

$$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} \times \begin{bmatrix} x_r & x_s \\ y_r & y_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- From the transformation and the definition, the covariant vectors are

$$\boldsymbol{g}_1 = (x_r, y_r) = \frac{\boldsymbol{v}^2 - \boldsymbol{v}^1}{2}, \quad \boldsymbol{g}_2 = (x_s, y_s) = \frac{\boldsymbol{v}^3 - \boldsymbol{v}^1}{2}.$$

  Using the result from the first question we have

$$\boldsymbol{g}^1 = (r_x, r_y) = \frac{1}{J}(y_s, -x_s), \quad \boldsymbol{g}^2 = (s_x, s_y) = \frac{1}{J}(-y_r, x_r),$$

  where

$$J = x_r y_s - x_s y_r.$$

- Substitution of $(r, s) = (0, 1/2)$ into the transformation gives $(x^*, y^*) = (1/2, 5/4)$. Similarly we have

$$\boldsymbol{g}_1 = (1/2, 1/2), \quad \boldsymbol{g}_2 = (0, 1/2), \quad J = 1/4$$

  and

$$\boldsymbol{g}^1 = (2, 0), \boldsymbol{g}^2 = (-2, 2).$$

5. Solve for the temperature $u\,(r, \theta, t)$ of the unsteady conduction problem in a circular disk

$$\alpha^2 \nabla^2 u = u_t \quad \text{in} \quad 0 \le r < a, \quad 0 < t < \infty$$

$$u(r, \theta, 0) = u_0, \quad u(a, \theta, t) = u_0 + u_0 \cos\theta$$

for $u(r, \theta, t)$. Expansion coefficients can be left undetermined.

Note: The following fact may be useful. The solution of the nth-order Bessel's differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

is given by $y(x) = AJ_n(x) + BY_n(x)$ where A, B are constants and J_n, Y_n are Bessel functions of first and second kind respectively. Also $Y_n \to -\infty$ as $x \to 0$.

Solution:

It is exercise 26.28c in the book, the solution is already provided.

6. Compute
$$I\left(t\right)=\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{st}}{s^{2}+b^{2}}ds,$$
where $\sigma>0$ and $b\in\mathbb{R}$ is given.

Solution:

It is exercise 15.11 in the book, the solution is already provided.

CSE Area A Exam 2020

1. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$ where $a_k \in A$ and the elements $a_1, \ldots, a_n$ need not be distinct. Prove that B_n is countable.

2. Let $T \in L(V, W)$, where V and W have inner products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$, respectively. Let T^* denote the adjoint of T.

   - Define the adjoint, show that it is a linear transformation, and that it is unique.
   - Suppose $V = \mathbb{R}^2$ with inner product $(\mathbf{x}, \mathbf{y})_V = x_1y_1 + 2x_2y_2$ and $W = \mathbb{R}^3$ with inner product $(\mathbf{u}, \mathbf{v})_W = u_1v_1 + u_2v_2 + u_3v_3$. Suppose T has matrix representation in the standard basis

   $$T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}$$

   Find the adjoint matrix $T^* : W \to V$.

3. Let $f : X \to Y$ be a bijection and $S \subset \mathcal{P}(X)$ a $\sigma-$algebra. Prove that $f(S) = \{f(A) : A \in S\}$ is a $\sigma-$algebra in Y.

4 Figure 1 presents a mapping between a reference triangle ($\boldsymbol{I}$) in $\boldsymbol{\xi} = (r, s)$-coordinates (computational) and an arbitrary triangle ($\boldsymbol{D}$) described by three given vertices $\boldsymbol{v}^1 = (x^1, y^1)$, $\boldsymbol{v}^2 = (x^2, y^2)$, $\boldsymbol{v}^3 = (x^3, y^3)$ in the $\boldsymbol{x} = (x, y)$-coordinates.

This mapping is pervasive in numerical methods for partial differential equations, especially the finite element methods. For this case, the map from computational domain $\boldsymbol{I}$ to the physical domain $\boldsymbol{D}$ is given as

$$\boldsymbol{x} = (x, y) = \boldsymbol{\Psi}(r, s) = -\frac{r+s}{2}\boldsymbol{v}^1 + \frac{r+1}{2}\boldsymbol{v}^2 + \frac{s+1}{2}\boldsymbol{v}^3$$

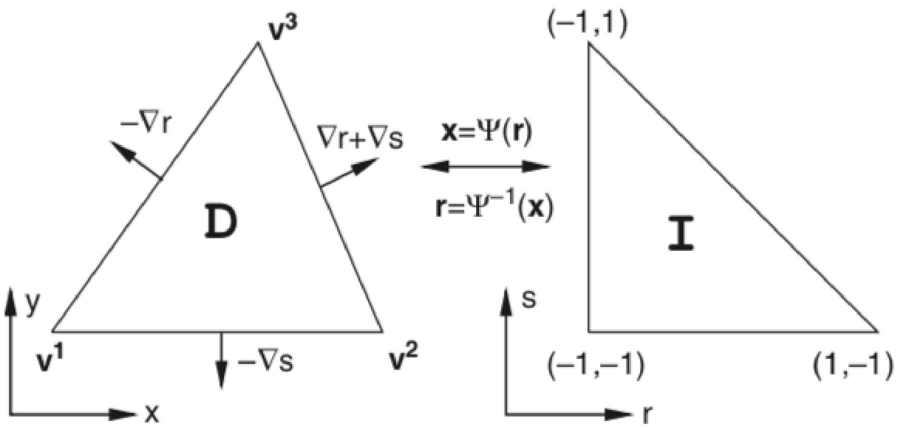


Figure 1: Mapping between computational triangle in the (r, s)-coordinates and the physical triangle in the (x, y) coordinates.

- Express the covariant and contravariant vectors in terms of $\boldsymbol{v}^k$, $k = 1, 2, 3$.
- Let $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ be a vector field in the physical domain $\boldsymbol{D}$. Recall the divergence of $\boldsymbol{u}$ in the physical domain is given by

  $$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}.$$

  Let

  $$\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{u} = \frac{\partial u_1}{\partial r} + \frac{\partial u_2}{\partial s}$$

  be the divergence of $\boldsymbol{u}$ in the computational domain.

  Derive an identity relating $\nabla_{\boldsymbol{\xi}} \cdot \boldsymbol{u}$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}$.

- Let $\boldsymbol{v}^1 = (0,0)\,, \boldsymbol{v}^2 = (1,1)\,, \boldsymbol{v}^3 = (0,1)$. Compute the area of $\boldsymbol{D}$ using the above parametrization and the computational domain $\boldsymbol{I}$.

5 Solve the following problem:

$$y_{tt} = a y_{xx}, \qquad -\infty < x < \infty$$
$$0 < t < \infty$$
$$a = \begin{cases} a_1 & x < 0 \\ a_2 & x \geq 0 \end{cases}$$

Consider an initial wave on the right, given by a function $F(x)$, which travels to the left (*see Fig. 2*). **Find the displacement of the string**.

What is the solution if $a_2 = 0$?

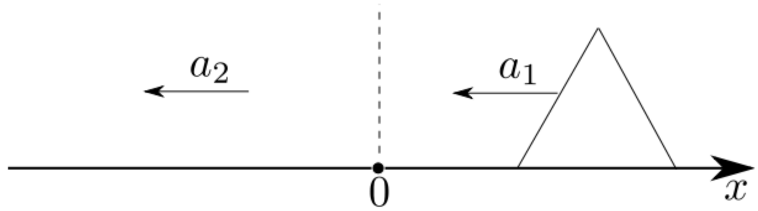
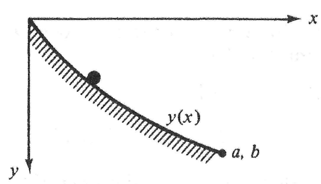


Figure 2: Wave traveling on an infinite string with non-uniform material.

6 *We seek the curve $y(x)$ from $(0,0)$ to (a,b) along which a bead of mass m will descend under the action of gravity (no friction) in the shortest time (see figure). Note that from conservation of energy,*



the bead's velocity is $v = \sqrt{2gy}$. Show that the variational problem may be stated as: Find $y(x)$ that mininizes

$$\int_0^a \sqrt{\frac{1+(y')^2}{2gy}}\,dx$$

subject to

$$y(0) = 0, y(a) = b.$$

Find the Euler equation and integrate the Euler equation once to show that

$$y' = \sqrt{\frac{1-Cy}{Cy}}.$$

Hint: *To show this you may need to show the following first:*

If $f = f(y, y')$ does not depend explicitly on x, then the Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] = 0$$

admits the first integral

$$f - y'\frac{\partial f}{\partial y'} = \text{constant.}$$

Area A-CSE
2021

CSEM Area A-CSE Preliminary Exam 2021

1. Consider rotation around the y-axis by an angle θ in the three-dimensional space depicted in Fig. 1.

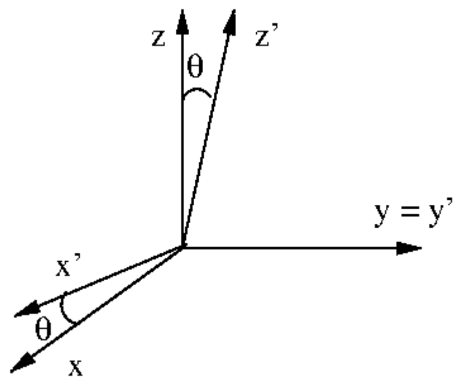


Figure 1: Rotation about the y-axis.

- Write down the explicit formula for the rotation as a map $R : \mathbb{R}^3 \to \mathbb{R}^3$.
- Prove that R is a linear map.
- Determine matrix representation of map R in the canonical basis $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$.
- Consider the canonical inner product in $\mathbb{R}^3$,

$$(x, y) = x_1y_1 + x_2y_2 + x_3y_3 \,,$$

  and determine the adjoint map to R with respect to this inner product. Is R an isometry ? Explain.
- Compute the adjoint of R with respect to a different, weighted inner product:

$$(x, y)_w = 3x_1y_1 + 2x_2y_2 + x_3y_3 \,.$$

  Is R an isometry with respect to this new inner product ?

Solution:

- $$R(x_1, x_2, x_3) = (x_1 \cos\theta - x_3 \sin\theta, x_2, x_1 \sin\theta + x_3 \cos\theta) \,.$$

- This is straightforward and, in particular, follows immediately from the following matrix representation.
- $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$
- Matrix representation of the adjoint R^* with respect to (wrt) the canonical inner product coincides with the transpose of the matrix representation of R and it represents the same rotation but by angle $-\theta$. R is an isometry wrt to the canonical inner product as it preserves the Euckidean norm (the length of vector).
- We need to have,
$$(Rx, y)_w = (x, R^*y)_w \,.$$
Accordingly,
$$\begin{aligned} (Rx, y)_w &= 3(x_1 \cos\theta - x_3 \sin\theta)y_1 + 2x_2y_2 + (x_1 \sin\theta + x_3 \cos\theta)y_3 \\ &= 3x_1(\cos\theta y_1 + \tfrac{1}{3}\sin\theta y_3) + 2x_2y_2 + x_3(-3\sin\theta y_1 + \cos\theta y_3) \end{aligned}$$
gives:
$$R^*y = (\cos\theta y_1 + \frac{1}{3}\sin\theta y_3, y_2, -3\sin\theta y_1 + \cos\theta y_3)\,.$$
R is no longer an isometry wrt the weighted inner product. It is easy to check that
$$\|Rx\|_w^2 - \|x\|_w^2 = (Rx, Rx)_w - (x, x)_w \neq 0\,.$$

2. Show that any two norms $\|\cdot\|_p, \|\cdot\|_q$, $1 \leq p, q \leq \infty$ in space $\mathbb{R}^n$ are equivalent, i.e. there exist constants C_1, C_2 such that

$$\|x\|_p \leq C_1 \|x\|_q \quad \text{and} \quad \|x\|_q \leq C_2 \|x\|_p \quad \forall x \in \mathbb{R}^n\,.$$

Solution: It is sufficient to show that any p-norm, for $p \in [1, \infty)$, is equivalent with the ∞-norm. We have,

$$|x_i| \leq (\sum_{j=1}^{n} |x_j|^p)^{\frac{1}{p}} =: \|x\|_p \quad i = 1, \ldots, n$$

and, therefore,

$$\|x\|_\infty := \max_{i=1,\ldots,n} |x_i| \leq \|x\|_p\,.$$

On the other side,

$$\begin{aligned}
\|x\|_p &= (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \\
&\leq (\sum_{i=1}^{n} \|x\|_\infty^p)^{\frac{1}{p}} = (n\|x\|_\infty^p)^{1/p} \\
&\leq n^{1/p} \|x\|_\infty\,.
\end{aligned}$$

3. Consider an initial-value problem:

$$\begin{cases} q \in C([0,T]) \cap C^1(0,T) \\ \dot{q} = t/ln(q) \\ q(0) = 2 \end{cases}$$

Use the Contractive Map Theorem to determine a concrete value of T for which the problem has a unique solution.

Solution: The problem is equivalent to the integral equation:

$$q(t) = 2 + \int_0^t \frac{s\,ds}{\ln q(s)}\,.$$

Consider the Chebyshev space $C([0,T])$ and a subset $B \subset C([0,T])$,

$$B := \{q \in C([0,T]) \,:\, q(t) \geq 2 \quad t \in [0,T]\}\,.$$

Define a nonlinear map,

$$A : B \to B \quad (Aq)(t) = 2 + \int_0^t \frac{s\,ds}{\ln q(s)}\,.$$

The map is well defined for any value of T, i.e., it takes set B into itself. This follows from the fact that for $q \in B$, the integrand is positive and, therefore, $(Aq)(t) \geq 2$.

We invesitigate now under what additional condition, map A is a contraction. Consider function,

$$F(q) = \frac{1}{\ln q}, \quad q \geq 2$$

We have,

$$|\frac{dF}{dq}| = |-\frac{1}{q\ln^2 q}| \leq \frac{1}{2\ln^2 2} \quad \text{for } q \geq 2\,.$$

Consequently, by the mean-value theorem, function $F(q)$ is Lipshitz continuous,

$$|F(q_1) - F(q_2)| = |F'(c)(q_1 - q_2)| \leq |F'(c)|\,|q_1 - q_2| \leq \frac{1}{2\ln^2 2}|q_1 - q_2|\,, \quad \text{for } 2 \leq q_2 < q_1\,.$$

This gives:

$$\begin{aligned} |(Aq_1)(t) - (Aq_2)(t)| &= |\int_0^t \left(\frac{1}{\ln q_1(s)} - \frac{1}{\ln q_2(s)}\right) ds| \\ &\leq \int_0^t s\frac{1}{2\ln^2 2}(q_1(s) - q_2(s))\,ds \\ &\leq \frac{t}{2\ln^2 2}\|q_1 - q_2\|_\infty\,. \end{aligned}$$

Consequently,

$$\|(Aq_1) - (Aq_2)\|_\infty = \sup_{t\in[0,T]} |(Aq_1)(t) - (Aq_2)(t)| \leq \frac{T}{2\ln^2 2}\|q_1 - q_2\|_\infty\,.$$

Therefore, for any $T < 2\ln^2 2$, map $A : B \to B$ is a contraction and, by the Banach Contractive Map Theorem, the integral equation has a unique solution.

4. Fig. 2 presents a mapping between a reference triangle ($\boldsymbol{I}$) in $\boldsymbol{\xi} = (r, s)$-coordinates (computational) and an arbitrary triangle ($\boldsymbol{D}$) described by three given vertices $\boldsymbol{v}^1 = \left(x^1, y^1\right), \boldsymbol{v}^2 = \left(x^2, y^2\right), \boldsymbol{v}^3 = \left(x^3, y^3\right)$ in the $\boldsymbol{x} = (x, y)$-coordinates.

   This mapping is pervasive in numerical methods for partial differential equations, especially the finite element methods. For this case, the map from computational domain $\boldsymbol{I}$ to the physical domain $\boldsymbol{D}$ is given as

$$\boldsymbol{x} = (x, y) = \boldsymbol{\Psi}(r, s) = -\frac{r+s}{2}\boldsymbol{v}^1 + \frac{r+1}{2}\boldsymbol{v}^2 + \frac{s+1}{2}\boldsymbol{v}^3$$

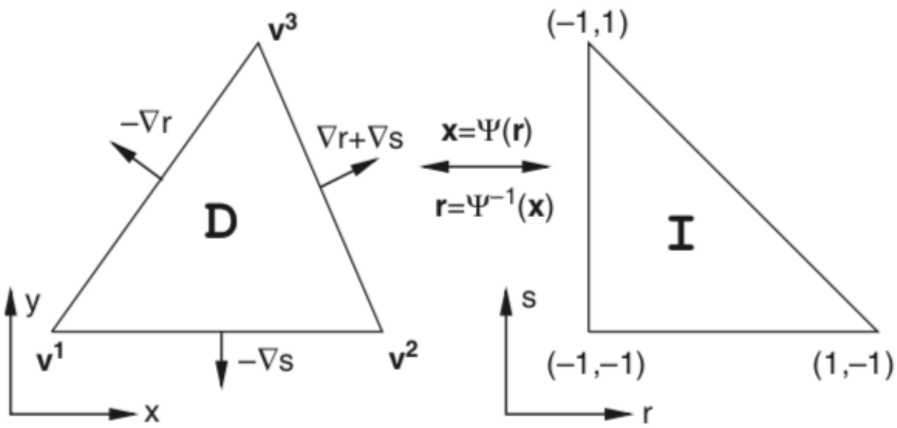


Figure 2: Mapping between computational triangle in the (r, s)-coordinates and the physical triangle in the (x, y) coordinates.

- Let $\boldsymbol{g}_j$, $j = 1, 2$ and $\boldsymbol{g}^i$, $i = 1, 2$ be covariant vectors and contravariant vectors, respectively. Express $\boldsymbol{g}_j$, $j = 1, 2$ in terms of $\boldsymbol{g}^i$, $i = 1, 2$, and the Jacobian $J = \det[\boldsymbol{g}_1, \boldsymbol{g}_2]$, where det denotes the determinant and $[\boldsymbol{g}_1, \boldsymbol{g}_2]$ is the matrix whose columns are the covariant vectors.
- Express the covariant and contravariant vectors in terms of $\boldsymbol{v}^k$, $k = 1, 2, 3$.
- Consider the following divergence free equation (modeling incompressible flow with flow field $\boldsymbol{u}(\boldsymbol{x})$):

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}(\boldsymbol{x}) = 0, \quad \text{in } \boldsymbol{D}, \tag{0.1}$$

  where $\nabla_{\boldsymbol{x}} \cdot$ denotes the divergence operator in the $\boldsymbol{s}$, i.e.,

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}(\boldsymbol{x}) = \frac{\partial \boldsymbol{u}}{\partial x} + \frac{\partial \boldsymbol{u}}{\partial y}. \quad = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}$$

  We are interested in expressing equation (0.1) in the reference domain $\boldsymbol{I}$ as it is "typically easier" to solve. Derive the equivalent equation in the references domain in terms of $\frac{\partial \boldsymbol{u}}{\partial r}, \frac{\partial \boldsymbol{u}}{\partial s}, \boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^3$, and J.

  $\frac{\partial u_1}{\partial r}, \frac{\partial u_2}{\partial s}$

Solution:

- See in lecture notes, in particular:

$$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} \times \begin{bmatrix} x_r & x_s \\ y_r & y_s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- From the transformation and the definition, the covariant vectors are

$$\boldsymbol{g}_1 = (x_r, y_r) = \frac{\boldsymbol{v}^2 - \boldsymbol{v}^1}{2}, \quad \boldsymbol{g}_2 = (x_s, y_s) = \frac{\boldsymbol{v}^3 - \boldsymbol{v}^1}{2}.$$

  Using the result from the first question we have

$$\boldsymbol{g}^1 = (r_x, r_y) = \frac{1}{J}\left(y_s, -x_s\right), \quad \boldsymbol{g}^2 = (s_x, s_y) = \frac{1}{J}\left(-y_r, x_r\right),$$

  where

$$J = x_r y_s - x_s y_r.$$

- Substitution of $(r, s) = (0, 1/2)$ into the transformation gives $(x^*, y^*) = (1/2, 5/4)$. Similarly we have

$$\boldsymbol{g}_1 = (1/2, 1/2), \quad \boldsymbol{g}_2 = (0, 1/2), \quad J = 1/4$$

  and

$$\boldsymbol{g}^1 = (2, 0), \quad \boldsymbol{g}^2 = (-2, 2).$$

5. Consider the following partial differential equation.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad \text{in } (0, \ell)\,, \\ u(0,t) &= u_L, \\ u(\ell,t) &= u_R, \\ u(x,0) &= u_0(x)\,,\end{aligned}$$

where α, ℓ, u_L, u_R are given numbers and $u_0(x)$ is a given function of x. Determine a solution to the initial-boundary-value problem. Show that the solution you have found is unique. (**Hint**: To show the uniqueness, the energy method discussed in class may be useful.)

Solution:

It is exercise 26.28c in the book, the solution is already provided.

6. Consider the variational problem over a square domain $\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and $w \in \mathbf{C}^1(\Omega)$:

$$\min_w I(w), \quad I(w) = \int_\Omega \left[\frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] dxdy$$
$$\text{subject to} \quad w|_{\partial \Omega} = f(x, y)$$

- Derive the Euler-Lagrange equation for the variational problem.
- Find a minimizer for $f(x, y) = 0$. Is it unique ? (**Hint**: To show the uniqueness, the energy method that we discussed in class may be useful.)

Solution:

It is exercise 15.11 in the book, the solution is already provided.

CSEM Area A-CSE Preliminary Exam 2022

1. A linear algebra "sanity check".

   Consider $\mathbb{R}^3$. Let A be the mirror (symmetry) transformation with respect to plane $x_1 + x_2 = 0$ (think about a mirror placed in this position and the transformation that maps a point P into its mirror image P').

   (a) Is A a linear map ? Explain.

   (b) Write down the matrix representation for map A in the canonical basis.

   (c) Explain why all linear maps from $\mathbb{R}^3$ into itself - $L(\mathbb{R}^3, \mathbb{R}^3)$, form a vector space. What is the dimension of the space ?

   (d) Do the mirror transformations (with respect an arbitrary plane passing through the origin) form a vector subspace of $L(\mathbb{R}^3, \mathbb{R}^3)$? Explain your answer. If yes, what is the dimension of this subspace ?

   (e) Define adjoint for a linear operator in a general Hilbert setting.

   (f) Compute the adjoint of map A with respect to the canonical inner product in $\mathbb{R}^3$. Is A a self-adjoint map ?

   (g) Define an orthonormal matrix.

   (h) Is matrix representation of map A (any mirror transformation) an orthonormal matrix ? Explain, why ?

   (20 points)

   Answers:

   (a) Yes, it is. Operations of taking a mirror image and multiplication by a number, commute. Similarly, vector addition and the mirror image map commute as well.

   (b) This is really a 2D problem. By inspection, the mirror map is:

   $$(x_1, x_2, x_3) \to (-x_2, -x_1, x_3)$$

   or,

   $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

(c) Functions defined on any set (in our case $\mathbb{R}^3$) with values in a vector space (in our case $\mathbb{R}^3$) equipped with pointwise addition and scalar multiplication, form a vector space. One has only to argue that the linear maps form a subset closed with respect to the vector space operations and, therefore, form a vector subspace of all functions defined on $\mathbb{R}^3$. This follows from the fact that a linear combination of linear maps is a linear map itself. Dimension of $L(X, Y)$ is always equal to the product of dim $X = n$ and dim $Y = m$ (in our case = 9). This follows from the isomorphism between $L(X, Y)$ and $m \times n$ matrices.

(d) Well, they do not. One possible way to see this, is to develop the matrix representation for a general mirror map corresponding to an arbitrary plane,

$$a_1x_1 + a_2x_2 + a_3x_3 = 0, \quad a_1^2 + a_2^2 + a_3^2 = 1 \,.$$

Equation for a straight line passing through point (x_1, x_2, x_3) and orthogonal to the plane, is:

$$\begin{aligned} y_1 &= x_1 + \lambda a_1 \\ y_2 &= x_2 + \lambda a_2 \\ y_3 &= x_3 + \lambda a_3 \end{aligned}$$

The value of parameter λ corresponding to the intersection point with the plane is obtained by plugging the formulas for y_i into the equation of the plane, we get:

$$\lambda = -(a_1x_1 + a_2x_2 + a_3x_3) \,.$$

The mirror image of point x is obtained now by doubling the value of the parameter,

$$\begin{aligned} y_1 &= x_1 - 2(a_1x_1 + a_2x_2 + a_3x_3)a_1 \\ y_2 &= x_2 - 2(a_1x_1 + a_2x_2 + a_3x_3)a_2 \\ y_3 &= x_3 - 2(a_1x_1 + a_2x_2 + a_3x_3)a_3 \end{aligned}$$

The corresponding matrix representation is:

$$\begin{pmatrix} 1 - 2a_1^2 & -2a_1a_2 & -2a_1a_3 \\ -2a_1a_2 & 1 - 2a_2^2 & -2a_2a_3 \\ -2a_1a_3 & -2a_2a_3 & 1 - 2a_3^2 \end{pmatrix}$$

Due to the fact that coefficients a_i come from a unit sphere in $\mathbb{R}^3$, and the nonlinear dependence of the matrix wrt to the coefficients, it is easy to see that maps of this type do not form a vector space. More formally, we can argue that such matrices do not form a set closed with respect to multiplication by a scalar. Indeed, take the first column for our case, i.e. for $a_1 = a_2 = 1/\sqrt{2}, a_3 = 0$, and multiply it by a factor of two to obtain $(0, -2, 0)^T$. Can we find a plane, i.e. coefficients a_i such that the general formula will yield these values, i.e.,

$$1 - 2a_1^2 = 0 \quad - 2a_1a_2 = 2 \quad - 2a_1a_3 = 0 \,?$$

We get $a_1 = \pm 1/\sqrt{2}, a_3 = 0$ and then $a_2 = \mp \sqrt{2}$. But a_i must represent components of a unit vector, so the value for a_2 is not acceptable.

There are many other ways to convince yourself that the mirror images do not form a vector space.

(e) The notion of the adjoint involves two Hilbert spaces X and Y with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$. Given a linear map $A : X \to Y$, we define the adjoint map $A^* : Y \to X$ by:

$$A^* = R_X^{-1} A^T R_Y$$

where $A : Y^* \to X^*$ is the transpose of A, and R_X, R_Y are Riesz maps for X and Y, resp. Equivalently,

$$(Ax, y)_Y = (x, A^* y)_X \quad x \in X,\, y \in Y \,.$$

(f) Nothing to compute. The matrix is symmetric so the operator is self-adjoint, i.e. $A^* = A$.

(g) Matrix A is *orthonormal* if $A^{-1} = A^T$.

(h) Yes, it is.

2. Compute inverse Laplace transform of function

$$f(s) = \frac{1}{s-a} \quad a > 0\,.$$

(a) State the Residue Theorem (5 points)

(b) State the Lebesgue Dominated Convergence Theorem (5 points)

(c) Use the theorems to provide all necessary details. (10 points)

Solution:

(a) See the lecture notes.

(b) See the book.

(c) The inverse Laplace of transform of $f(s)$ is:

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{s-a}\,ds\,.$$

Use the integration contour shown in Fig. 1. The integrand has a single simple pole at $s = a$. The integral over the closed contour,

$$\frac{1}{2\pi i}\int \frac{e^{st}}{s-a}\,ds = \mathrm{Res}_a = \lim_{s\to a} e^{st} = e^{at}\,.$$

The integral over vertical part c converges to the integral in the inverse Laplace transform as $R\to\infty$. We need to demonstrate that the integrals over parts c_1, c_2, c_R vanish in the limit. The

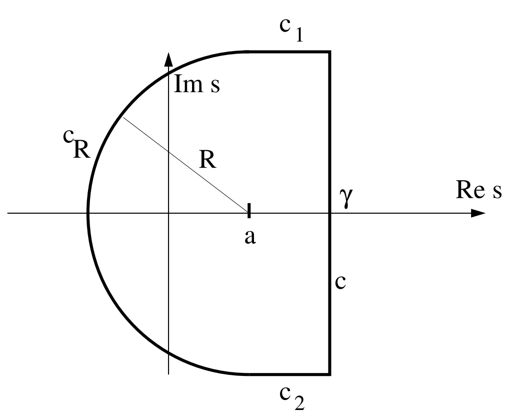


Figure 1: Integration contour.

parametrization for c_1 is:

$$z = (\gamma - \xi, R), \quad \xi \in (0, \gamma - a) \qquad dz = -d\xi\,,$$

and the integral over c_1 is estimated as follows:

$$|\int_{c_1} \frac{e^{st}}{s-a}\,ds| = |-\int_0^{\gamma-\xi} \frac{e^{(\gamma-\xi)t}e^{iRt}}{z-a}\,d\xi| \leq \frac{e^{\gamma t}}{R}\int_0^{\gamma-\xi} d\xi \to 0 \quad \text{as } R \to \infty\,.$$

By the same argument, intgeral over c_2 converges to zero as well. The parametrization for the circular part c_R is:

$$z = (a - R\sin\theta, R\cos\theta), \quad \theta \in (0,\pi) \qquad dz = Rd\theta\,.$$

The integral over c_R is estimates as follows,

$$|\int_{c_R} \frac{e^{st}}{s-a}\,ds| \leq \int_0^{\pi} \frac{e^{(a-R\sin\theta)t}}{R}\,Rd\theta = e^{at}\int_0^{\pi} e^{-R\sin\theta t}\,d\theta\,.$$

As the integrand converges a.e. pointwise to zero, and it has an integrable upper bound (a constant), the Lebesgue Theorem implies that the integral converges to zero.

3. Consider an initial-value problem:

$$\begin{cases} q \in C([0,T]) \cap C^1(0,T) \\ \dot{q} = \frac{q^2}{t-2} \\ q(0) = 1 \end{cases}$$

(a) Solve the problem by elementary means. (5 points)

(b) State (precisely) the Contractive Map Theorem. (2 points)

(c) Use the theorem todetermine a concrete value of T for which the problem has a unique solution. Compare with (a). (13 points)

Solution:

(a) Use separation of variables:

$$\frac{dq}{q^2} = \frac{dt}{t-2} \qquad \Rightarrow \qquad -\frac{1}{q}|_1^{q(t)} = \ln|t-2||_0^t \,.$$

We obtain:

$$q(t) = (\ln 2 + 1 - \ln|t-2|)^{-1} \,.$$

(b) See the book.

(c) The problem is equivalent to the integral equation:

$$q(t) = 1 + \int_0^t \frac{q^2(s)}{s-2}\, ds \,.$$

Consider the Chebyshev space $C([0,T])$ where T is to be determined. We need to identify a subset $B \subset C([0,T])$ such that the nonlinear map

$$A : B \to B \quad (Aq)(t) = 1 + \int_0^t \frac{q^2(s)}{s-2}\, ds$$

is, first of all, well-defined. As $(Aq)(0) = 1$, it is natural to define the set as

$$B := \{q \in C([0,T]) \,:\, |q(t) - 1| \le 1 \quad t \in [0,T]\} \,.$$

As a closed subset of a complete space, set B is complete as well. Constant 1 used in the bound is somehow arbitrary.

Step 1: We first determine T that guarantees that A maps set B into itself. First of all, if the integrand is bounded then Aq is Lipschitz continuous as,

$$|(Aq)(t_2) - (Aq)(t_1)| \le \int_{t_1}^{t_2} \underbrace{|\frac{q^2(s)}{s-2}|}_{\le C}\, ds \le C(t_2 - t_1) \,.$$

We now estimate,

$$|(Aq)(t) - 1| = |\int_0^t \frac{q^2(s)}{s-2}\, ds| \le \int_0^t \frac{q^2(t)}{|s-2|}\, ds \le \int_0^t \frac{4}{|s-2|}\, ds \le 4T$$

provided

$$\frac{1}{|s-2|} \le 1 \quad \Leftrightarrow \quad |s-2| \ge 1 \quad \Leftrightarrow \quad s-2 \ge 1 \text{ or } s-2 \le -1 \quad \Leftrightarrow \quad s \ge 3 \text{ or } s \le 1\,,$$

i.e., we assume $T \le 1$. The map is then well-defined if $4T \le 1$, i.e., $T \le \frac{1}{4}$.

Step 2: We check, under what condition on T, map A is contractive. We have,

$$\begin{aligned} |(Aq_1)(t) - (Aq_2)(t)| &\le \int_0^t \frac{|(q_1(s))^2 - (q_2(s))^2|}{|s-2|}\, ds \\ &= \int_0^t \frac{|(q_1(s) - q_2(s))(q_1(s) + q_2(s))|}{|s-2|}\, ds \\ &\le T4 \max_{t \in [0,T]} |q_1(t) - q_2(t)|\,. \end{aligned}$$

The map is thus a contraction if $4T < 1$, i.e. $T < \frac{1}{4}$. In cocnlusion, for any $T < \frac{1}{4}$, map $A : B \to B$ is a well-defined contraction and, by the Banach Contractive Map Theorem, the integral equation has a unique solution.

Comparing with solution from (a), we see that our estimate is quite conservative. The solution exists for $T < 2 + e$, it blows up at $T = 2 + e$.

4. Separation of variables. Solve the Laplace equation in a circular domain with a Dirichlet BC:

$$\begin{cases} -\Delta u &= 0 & r < 2 \\ u &= \cos(2\theta) & r = 2,\ \theta \in [0, 2\pi) \end{cases}$$

Explain precisely the relation with Sturm-Liouville Theorem and deliver the formula for the final solution. (20 points)

Solution:

$$-\Delta u = -\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0\,.$$

Assuming $u = R(r)\Theta(\theta)$, and separating variables, we get:

$$-\frac{\Theta''}{\Theta} = r(rR')' = \lambda\,.$$

The operator in θ, accompanied with periodic BCs:

$$\Theta(0) = \Theta(2\pi) \quad \Theta'(0) = \Theta'(2\pi)\,,$$

is self-adjoint and semi-positive definite, so we can assume $\lambda = k^2, k \geq 0$. Solution of the Sturm-Liouville problem in θ leads to:

$$\Theta = \begin{cases} A & k = 0 \\ A\cos k\theta + B\sin k\theta & k > 0\,. \end{cases}$$

In order to satisfy the periodic BCs, positive k's must be integers. The corresponding solutions $R(r)$ are:

$$R(r) = \begin{cases} C\ln r + D & k = 0 \\ Cr^{-k} + Dr^k & k > 0\,. \end{cases}$$

Functions $\ln r$ and r^{-k} are singular at zero and are eliminated. Using the superposition, we get the general solution in the form:

$$u = A_0 + \sum_{k=1}^{\infty} r^k(A_k\cos k\theta + B_k\sin k\theta)\,.$$

By the Sturm-Liouville Theorem, functions $1, \cos k\theta, \sin k\theta$ provide an L^2-orthogonal basis in θ. As the BC data: $\cos 2\theta$ is just one of them, the final solution will reduce to the single term:

$$u = \frac{1}{4}r^2\cos 2\theta\,.$$

Area A-CSE
May 2022
w/ solutions

5. Calculus of variations.

Consider the Euler-Bernoulli beam problem shown in Fig. 2. The elastic energy of the beam is given by:

$$\frac{1}{2}\int_0^l EI(w'')^2\,dx$$

where EI is the stiffness of the beam. Write down the work of the external forces, and the total potential energy functional. Specify the set (space) of kinematically admissble displacements, i.e., formulate the essential BCs, and write down the energy minimization problem. Write down then (derive ?) the corresponding variational formulation (Principle of Virtual Work). Use integration by parts and the Fourier argument to derive the corresponding E-L equation, and natural boundary conditions.

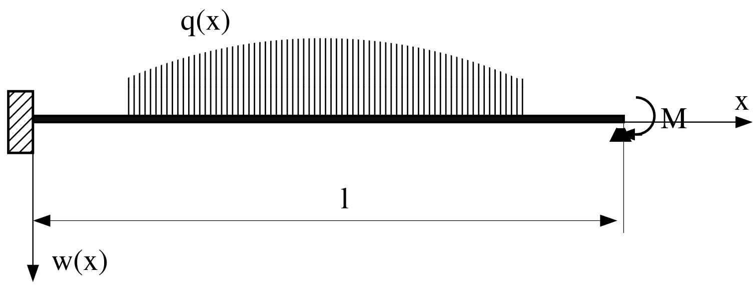


Figure 2: A beam problem. The beam is fixed at $x = 0$ and free supported at $x = l$, subjected to a distributed load with intensity $q = q(x)$, and a concentrated moment M at $x = l$.

(20 points)

Solution: The total potential energy functional is:

$$J(w) = \frac{1}{2}\int_0^l EI(w'')^2\,dx - \underbrace{(\int_0^l qw\,dx + Mw'(l))}_{\text{work of external forces}}\,.$$

The *space of kinematically admissible displacements* involves energy setting and essential BCs:

$$V := \{w \in H^2(0,l)\,:\, w(0) = w'(0) = w(l) = 0\}\,.$$

The minimization problem reads:

$$\begin{cases} \text{Find } w \in V \text{ such that:} \\ J(w) = \min_{z \in V} J(z)\,. \end{cases}$$

The corresponding variational formulation is:

$$\begin{cases} \text{Find } w \in V \text{ such that:} \\ EI\int_0^l w''v''\,dx = \int_0^l qv\,dx + Mv'(l) \qquad \forall v \in V\,. \end{cases}$$

Integrating twice by parts, we obtain:

$$EI \int_0^l w''''v\,dx - EIw'''v|_0^l + EIw''v'|_0^l = \int_0^l qv\,dx + Mv'(l) \qquad \forall v \in V\,.$$

Taking into account the BCs for test function v, we have:

$$EI \int_0^l w''''v\,dx + EIw''(l)v'(l) = \int_0^l qv\,dx + Mv'(l) \qquad \forall v \in V\,.$$

Fourier argument:

Step 1: Use $v \in V$ that satisfy an additional BC: $v''(l) = 0$ and use Fourier Lemma to obtain the E-L equation:

$$EIw'''' = q\,.$$

Step 2: The variational equation reduces then to:

$$EIw''(l)v'(l) = Mv'(l) \quad \forall v \in V\,.$$

This yields the natural BC:

$$EIw''(l) = M\,.$$

EXAM #____

CSEM Area A-CSE Preliminary Exam 2023

1. A linear algebra “sanity check”.

   Consider $\mathbb{R}^3$. Let A be the rotation about the x_2 axis with an angle of rotation $\theta > 0$, see Fig. 1.

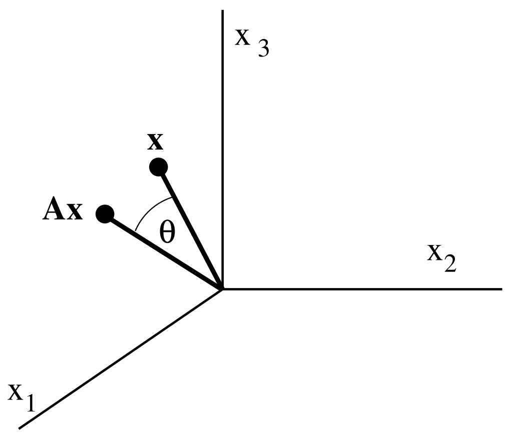


Figure 1: Rotation about x_2 axis.

   (a) Is A a linear map ? Explain.
   (b) Write down an explicit formula for A as a function from $\mathbb{R}^3$ into itself.
   (c) Write down the matrix representation for map A in the canonical basis.
   (d) Explain why all linear maps from $\mathbb{R}^3$ into itself - $L(\mathbb{R}^3, \mathbb{R}^3)$, form a vector space. What is the dimension of the space ?
   (e) Do the rotations around the x_2 axis with an arbitrary angle of rotation θ, form a vector subspace of $L(\mathbb{R}^3, \mathbb{R}^3)$? Explain your answer. If yes, what is the dimension of this subspace ?
   (f) Define adjoint for a linear operator in a general Hilbert setting.
   (g) Compute the adjoint of map A with respect to the canonical inner product in $\mathbb{R}^3$. Is A a self-adjoint map ?
   (h) Define an orthonormal matrix.
   (i) Is matrix representation of map A an orthonormal matrix ? Explain, why ?

   (20 points)

(20 points)

Answers:

(a) Yes, it is. Rotation A and multiplication by a number, commute. Similarly, vector addition and rotation A commute as well.

(b) Using linearity of R helps to derive the explicit formula for R,

$$\begin{aligned} Ax &= A(x_1e_1 + x_2e_2 + x_3e_3) \\ &= x_1Ae_1 + x_2Ae_2 + x_3Ae_3 = x_1(\cos\theta\, e_1 - \sin\theta\, e_3) + x_2e_2 + x_3(\sin\theta\, e_1 + \cos\theta\, e_3) \\ &= (\cos\theta\, x_1 + \sin\theta\, x_3)e_1 + x_2e_2 + (-\sin\theta\, x_1 + \cos\theta\, x_3)e_3\,. \end{aligned}$$

In other words,

$$A\,:\,(x_1, x_2, x_3) \to (\cos\theta\, x_1 + \sin\theta\, x_3, x_2, -\sin\theta\, x_1 + \cos\theta\, x_3)\,.$$

(c) Here it is:

$$A = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

(d) Functions defined on any set (in our case $\mathbb{R}^3$) with values in a vector space (in our case $\mathbb{R}^3$) equipped with pointwise addition and scalar multiplication, form a vector space. One has only to argue that the linear maps form a subset closed with respect to the vector space operations and, therefore, form a vector subspace of all functions defined on $\mathbb{R}^3$. This follows from the fact that a linear combination of linear maps is a linear map itself. Dimension of $L(X, Y)$ is always equal to the product of dim $X = n$ and dim $Y = m$ (in our case = 9). This follows from the isomorphism between $L(X, Y)$ and $m \times n$ matrices.

(e) Well, they do not. Composition of two such rotations is a rotation but the algebraic sum of two rotations is not a rotation. One way to see it is to notice that the rotation preserves length (is an isometry), i.e. $\|Rx\| = \|x|$, where $\|\cdot\|$ denotes the Euclidean norm of x. But the sum of two rotations does not and, therefore, it cannot be a rotation. It is worth mentioning though that rotations about infinitesimally small angles:

$$Rx = (x_1 + \theta x_3, x_2, -\theta x_1 + x_3) \qquad (\cos\theta \approx 1,\, \sin\theta \approx \theta)\,,$$

do form a one-dimensional linear subspace.

(f) The notion of the adjoint involves two Hilbert spaces X and Y with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$. Given a linear map $A : X \to Y$, we define the adjoint map $A^* : Y \to X$ by:

$$A^* = R_X^{-1} A^T R_Y$$

where $A : Y^* \to X^*$ is the transpose of A, and R_X, R_Y are Riesz maps for X and Y, resp. Equivalently,

$$(Ax, y)_Y = (x, A^*y)_X \quad x \in X,\, y \in Y\,.$$

(g) In the canonical basis, the matrix representation of A^* is the transpose of matrix representation of A, i.e.

$$A^* = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

As the two matrices are different, the operator is not self-adjoint.

(h) Matrix A is *orthonormal* if $A^{-1} = A^T$.

(i) Yes, it is. The inverse of rotation by angle θ is the rotation by angle $-\theta$. Its matrix representation is the transpose of the original matrix. In other words, $A^{-1} = A^*$.

2. Consider an initial-value problem:

$$\begin{cases} q \in C([0,T]) \cap C^1(0,T) \\ \dot{q} = \frac{q}{(t-1)^2} \\ q(0) = 1 \end{cases}$$

(a) Solve the problem by elementary means. (5 points)

(b) State (precisely) the Contractive Map Theorem. (2 points)

(c) Use the theorem to determine a concrete value of T for which the problem has a unique solution. Compare with (a). (13 points)

Solution:

(a) Use separation of variables:

$$\frac{dq}{q} = \frac{dt}{(t-1)^2} \qquad \Rightarrow \qquad \ln|q|\Big|_1^{q(t)} = -(t-1)^{-1}\Big|_0^t .$$

We obtain:

$$|q(t)| = \exp(-(t-1)^{-1} - 1) \qquad \Rightarrow \qquad q(t) = \pm \exp(-(t-1)^{-1} - 1) .$$

As $q(0) = 1$, we can eliminate the negative branch, and

$$q(t) = \exp(-(t-1)^{-1} - 1) .$$

Note that $q(t) \to \infty$ as $t \to 1_-$.

(b) See the book.

(c) The problem is equivalent to the integral equation:

$$q(t) = 1 + \int_0^t \frac{q(s)}{(s-1)^2} \, ds .$$

Consider the Chebyshev space $C([0,T])$ where T is to be determined. We need to identify a subset $B \subset C([0,T])$ such that the nonlinear map

$$A : B \to B \quad (Aq)(t) = 1 + \int_0^t \frac{q(s)}{(s-1)^2} \, ds$$

is, first of all, well-defined. As $(Aq)(0) = 1$, it is natural to define the set as,

$$B := \{q \in C([0,T]) \, : \, |q(t) - 1| \le 1 \quad t \in [0,T]\} .$$

As a closed subset of a complete space, set B is complete as well. Constant 1 used in the bound is somehow arbitrary.

Step 1: We first determine T that guarantees that A maps set B into itself. First of all, if the integrand is bounded then Aq is Lipschitz continuous[1] as, for $t_1 < t_2$,

$$|(Aq)(t_2) - (Aq)(t_1)| \leq \int_{t_1}^{t_2} \underbrace{|\frac{q(s)}{(s-1)^2}|}_{\leq C} \, ds \leq C(t_2 - t_1)\,.$$

We now estimate,

$$\begin{aligned} |(Aq)(t) - 1| &= |\int_0^t \frac{q(s)}{(s-1)^2}\, ds| \leq \int_0^t \frac{|q(s)|}{(s-1)^2}\, ds \\ &\leq 2\int_0^t \frac{1}{(s-1)^2}\, ds & (|q(s)| \leq |q(s)-1| + 1 \leq 2) \\ &= -2(s-1)^{-1}|_0^t = 2\left[\frac{1}{1-t} - 1\right]. \end{aligned}$$

Requesting,

$$2\left[\frac{1}{1-t} - 1\right] \leq 1$$

we obtain: $t \leq \frac{1}{3}$.

Step 2: We check, under what condition on T, map A is contractive. We have,

$$\begin{aligned} |(Aq_1)(t) - (Aq_2)(t)| &\leq \int_0^t \frac{|q_1(s) - q_2(s)|}{|(s-1^2|}\, ds \\ &= \underbrace{\int_0^t \frac{ds}{|(s-1)|}\, ds}_{=\frac{1}{1-t}-1} \max_{t\in[0,T]} |q_1(t) - q_2(t)|\,. \end{aligned}$$

The map is thus a contraction if $\frac{1}{1-t} - 1 < 1$ which gives $t < \frac{1}{2}$. In conclusion, for any $T \leq \frac{1}{3}$, map $A : B \to B$ is a well-defined contraction and, by the Banach Contractive Map Theorem, the integral equation has a unique solution.

Comparing with solution from (a), we see that our estimate is quite conservative. The solution exists for $T < 1$, it blows up at $T = 1$.

[1]And, therefore, continuous.

3. Metric spaces.

   (a) Explain the difference between two *equivalent metrics* and *topologically equivalent metrics*. (5 points)

   (b) Let (X, d) be an arbitrary metric space. Prove that function

   $$\rho(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

   is also a metric on space X. (10 points)

   (c) Are the two metrics (in general) equivalent ? topogically equivalent ? (5 points)

Solution:

(a) See the book.

(b) See the book.

(c) In general, the two metrics cannot be equivalent. The second metric is bounded (by one), the first might not be bounded at all. But they are topologically equivalent, i.e., the corresponding bases of neighbohoods (the balls) are equivalent. Indeed, let

$$B^d(x, \epsilon) := \{y \in X \,:\, d(x, y) < \epsilon\}, \qquad B^\rho(x, \epsilon) := \{y \in X \,:\, \rho(x, y) < \epsilon\}\,.$$

Since

$$d(x, y) < \epsilon \quad \Rightarrow \quad \rho(x, y) < d(x, y) < \epsilon\,.$$

we have,

$$B^d(x, \epsilon) \subset B^\rho(x, \epsilon)\,.$$

Similarly, for an arbitrary $\epsilon > 0$, and $\epsilon_1 = \frac{\epsilon}{1+\epsilon}$,

$$B^\rho(x, \epsilon_1) \subset B^d(x, \epsilon)\,.$$

Indeed,

$$\rho = \frac{d}{1 + d} < \epsilon_1 \quad \Leftrightarrow \quad d < \frac{\epsilon_1}{1 - \epsilon_1} = \epsilon\,.$$

CSEM Qualifying: Second Part

May 15, 2023

Important: In order to get full grades, for every question, you need to provide the details of your work on how to get to a solution or the end of the proof.

Problem 1 (SVD: 10 points)**.** *Consider* $\mathscr{A} : \mathbb{U} = Span\{1,\ x\} \subset L^2(-1,\ 1) \mapsto \mathbb{R}^2$ *such that the map A defined as*

$$f(x) \in \mathbb{U} \mapsto \mathscr{A} f = \begin{bmatrix} \int_{-1}^{1} f(x)\, dx \\ \int_{-1}^{1} (2x+1) f(x)\, dx \end{bmatrix}$$

Find the singular triplets of $\mathscr{A}$*.*

You may need the following two eigenvalue decompositions:

$$\begin{bmatrix} 4 & 4 \\ 4 & 52/9 \end{bmatrix} = \begin{bmatrix} -0.78 & 0.62 \\ 0.62 & 0.78 \end{bmatrix} \begin{bmatrix} 0.79 & 0 \\ 0 & 8.98 \end{bmatrix} \begin{bmatrix} -0.78 & 0.62 \\ 0.62 & 0.78 \end{bmatrix},$$

and

$$\begin{bmatrix} 8 & 8/3 \\ 8/3 & 16/9 \end{bmatrix} = \begin{bmatrix} 0.34 & -0.93 \\ -0.93 & -0.34 \end{bmatrix} \begin{bmatrix} 0.79 & 0 \\ 0 & 8.98 \end{bmatrix} \begin{bmatrix} 0.34 & -0.93 \\ -0.93 & -0.34 \end{bmatrix}.$$

Solution 1. *This is a simplified version of a homework. We solve the problem in a couple of simple steps:*

- *The matrix representation A of the operator* $\mathscr{A}$ *in the orthogonormal bases of* $\mathbb{U}$ *and* $\mathbb{R}^2$ *is*

$$\begin{bmatrix} 2 & 0 \\ 2 & 4/3 \end{bmatrix}.$$

- *Since* $A^T A = \begin{bmatrix} 8 & 8/3 \\ 8/3 & 16/9 \end{bmatrix}$ *and* $AA^T = \begin{bmatrix} 4 & 4 \\ 4 & 52/9 \end{bmatrix}$, *the SVD of A is thus given as*

$$A = \begin{bmatrix} -0.93 & 0.34 \\ -0.34 & -0.93 \end{bmatrix} \begin{bmatrix} 2.99 & 0 \\ 0 & 0.89 \end{bmatrix} \begin{bmatrix} 0.62 & 0.78 \\ -0.78 & 0.62 \end{bmatrix}$$

The singular triplets of $\mathscr{A}$ *are thus*

$$\left\{ 2.99, 0.62 + 0.78x, \begin{bmatrix} -0.93 \\ -0.34 \end{bmatrix} \right\}$$

and

$$\left\{ 0.89, -0.78 + 0.62x, \begin{bmatrix} 0.34 \\ -0.93 \end{bmatrix} \right\}$$

Problem 2 (Optimization: 10 points)**.** *Consider the following optimization problem*

$$\min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x}^T A \mathbf{x}$$

subject to

$$\|\mathbf{x}\| = 1,$$

where $A \in \mathbb{R}^{n \times n}$ *is a symmetric matrix, and* $\|\cdot\|$ *is the standard Euclidean norm in* $\mathbb{R}^n$*. What is the optimal solution (derive and interpret it)?*

Solution 2. *The Lagrangian*

$$L(\mathbf{x}, \lambda) = \mathbf{x}^T A\mathbf{x} - \lambda\left(\mathbf{x}^T\mathbf{x} - 1\right)$$

and thus the first order optimality condition gives

$$A\mathbf{x} = \lambda\mathbf{x},$$

which implies

$$\mathbf{x}^T A\mathbf{x} = \lambda,$$

and thus the optimal solution is the eigenvector of A corresponding to the smallest eigenvalue.

Problem 3 (PDE: 10 points)**.** *Consider the following "PDE" in one dimension*

$$-\frac{d^2u}{dx^2} - \alpha u = f \text{ in } (-1,1),$$
$$u(-1) = u(1) = 0,$$

where α is a positive constant.

Choose (with reasoning why) an appropriate α so that (meaning prove that) with that appropriate value of α there exists a unique solution residing in $H_0^1 := \left\{v \in H^1(-1,1) : v(-1) = v(1) = 0\right\}$ to the problem for any $f \in L^2(-1,1)$ and that the unique solution u depends continuously on f.

You may need the following inequality

$$\int_{-1}^{1} \left|\frac{du}{dx}\right|^2 dx \ge \frac{1}{4}\int_{-1}^{1} |u|^2\, dx$$

Solution 3. *For any $v \in H_0^1$, and from the the bilinear form of the weak formulation we have: for any $0 < \varepsilon < 1$*

$$\left\|\frac{du}{dx}\right\|_{L^2}^2 - \alpha\|u\|_{L^2}^2 \ge \varepsilon\left\|\frac{du}{dx}\right\|_{L^2}^2 + \left(\frac{1-\varepsilon}{4} - \alpha\right)\|u\|_{L^2}^2 \ge \frac{\varepsilon}{2}\left\|\frac{du}{dx}\right\|_{L^2}^2 + \frac{\varepsilon}{8}\|u\|_{L^2}^2 + \left(\frac{1-\varepsilon}{4} - \alpha\right)\|u\|_{L^2}^2$$
$$\ge \frac{\varepsilon}{8}\|u\|_{H^1}^2 + \left(\frac{1-\varepsilon}{4} - \alpha\right)\|u\|_{L^2}^2,$$

where we have used the inequality in the hint (Poincare-Friedrichs) in the second inequality.

Thus, picking any value of $0 < \varepsilon < 1$ and setting $\alpha = \frac{1-\varepsilon}{4}$ will work. For example, taking $\varepsilon = 1/2$, and then taking $\alpha = 1/8$ will ensure the coercivity with coerivity constant $1/16$, and the problem is well-posed by Lax-Milgram.

CSEM Area A-CSE Preliminary Exam 2024

Solve the following five problems in 3 hours.

1. A linear algebra "sanity check".

   Consider $\mathbb{R}^3$. Let $a = (1, 2, 3)$. Consider the function:

$$\mathbb{R}^3 \ni x \to a \times x \in \mathbb{R}^3$$

   where $\times$ denotes the cross product.

   (a) Is A a linear map ? Explain.

   (b) Write down the matrix representation for map A in the canonical basis.

   (c) Explain why all linear maps from $\mathbb{R}^3$ into itself - $L(\mathbb{R}^3, \mathbb{R}^3)$, form a vector space. What is the dimension of the space ?

   (d) Do the maps defined as above but with different vectors a form a vector subspace of $L(\mathbb{R}^3, \mathbb{R}^3)$? Explain your answer. If yes, what is the dimension of this subspace ?

   (e) Define adjoint for a linear operator in a general Hilbert setting.

   (f) Compute the adjoint of map A with respect to the canonical inner product in $\mathbb{R}^3$. Is A a self-adjoint map ?

   (20 points)

   Answers:

   (a) Yes, it is. The cross product is a linear operation of its second argument,

$$a \times (\alpha x + \beta y) = \alpha(a \times x) + \beta(a \times y)\,.$$

   (b) Easy,

$$a \times x = (1, 2, 3) \times (x_1, x_1, x_3) = (2x_3 - 3x_2, 3x_1 - x_3, x_2 - 2x_1)\,.$$

   The matrix representaion in the canonical basis is:

$$\begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

(c) Functions defined on any set (in our case $\mathbb{R}^3$) with values in a vector space (in our case $\mathbb{R}^3$) equipped with pointwise addition and scalar multiplication, form a vector space. One has only to argue that the linear maps form a subset closed with respect to the vector space operations and, therefore, form a vector subspace of all functions defined on $\mathbb{R}^3$. This follows from the fact that a linear combination of linear maps is a linear map itself. Dimension of $L(X, Y)$ is always equal to the product of dim $X = n$ and dim $Y = m$ (in our case = 9). This follows from the isomorphism between $L(X, Y)$ and $m \times n$ matrices.

(d) Well, they do. This is a consequence of the fact that the cross product is also a linear function with respect to its first argument.

(e) The notion of the adjoint involves two Hilbert spaces X and Y with inner products $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_Y$. Given a linear map $A : X \to Y$, we define the adjoint map $A^* : Y \to X$ by:

$$A^* = R_X^{-1} A^T R_Y$$

where $A : Y^* \to X^*$ is the transpose of A, and R_X, R_Y are Riesz maps for X and Y, resp. Equivalently,

$$(Ax, y)_Y = (x, A^* y)_X \quad x \in X,\, y \in Y\,.$$

(f) Nothing to compute. The matrix of the adjoint operator is the transpose of the original matrix. In our case the matrix is *antisymmetric* so $A^* = -A$. The operator is *skew adjoint* but it is *not* self-adjoint.

2. Elementary ODEs, Dirac's delta, Laplace transform and the Residue Theorem.

   Consider the following initial-value problem.

$$\begin{cases} \ddot{x} - x = 2\delta(t-1) \\ x(0) = 0, \; \dot{x}(0) = 0 \end{cases}$$

   where δ denotes the Dirac's delta functional.

   (a) Define precisely delta functional and reinterpret its action on unknown function $x(t)$ in terms of appropriate jump conditions.

   (b) Solve the problem using elementary means.

   (c) Define the Laplace transform. Apply it to both sides of the equation and find the solution in the Laplace domain.

   (d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the Laplace domain and compare it with the solution obtained using the elementary calculus.

   (20 points)

   Solution:

   (a) Dirac's delta at $t = 1$ is a functional that assigns to every test function ϕ its value at $t = 1$,

$$\mathcal{D}(\mathbb{R}) \ni \phi \to \phi(1) \in \mathbb{R}$$

   Delta is the distributional derivative of the Heaviside function. Its presence translates into jump conditions at $t = 1$,

$$[x(1)] = 0, \quad [\dot{x}(1)] = 2\,.$$

   (b) For $t \in (0, 1)$,

$$x(t) = Ae^{t} + Be^{-t}$$

   Utilizing IC, we get

$$x(t) = 0$$

   For $t \in (1, \infty)$,

$$x(t) = Ae^{t-1} + Be^{-(t-1)}$$

   Utilizing jump conditions at $t = 1$ and the known value of x and $\dot{x}$ at $t = 1-$ (both equal zero), we get

$$x(t) = e^{t-1} + e^{-(t-1)}\,.$$

   (c)

$$\mathcal{L}(f)(s) = \bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$\int_0^\infty e^{-st} 2\delta(t-1)\, dt = 2e^{-s}$$

Recall the formulas resulting from integration by parts,

$$\bar{\dot{x}} = s\bar{x}(s) - x(0)$$

$$\bar{\ddot{x}} = s^2\bar{x}(s) - sx(0) - \dot{x}(0)$$

Transforming both sides of the equation and accounting for the IC, we get;

$$(s^2 - 1)\bar{x} = 2e^{-s}$$

which gives the solution in the Laplace domain,

$$\bar{x}(s) = 2\frac{e^{-s}}{(s-1)(s+1))} = \frac{e^{-s}}{s-1} - \frac{e^{-s}}{s+1}\,.$$

(d) The following is just a sketch, look up your lecture notes for a detailed explanation.

First term :

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{-s}e^{st}}{s-1}\, ds = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-1)}}{s-1}\, ds\,.$$

Case: $t < 1$. Use contour to the right to conclude that $x = 0$.

Case: $t > 1$. Simple pole at $s = 1$. Use contour to the left to conclude that

$$x = \text{Re}_1 \frac{e^{s(t-1)}}{s} = \lim_{s\to 1} e^{s(t-1)} = e^{t-1}\,.$$

The argument showing that the integral over C_R vanishes in the limit, needs the use of the Lebesgue Dominated Convergence Theorem.

Second term :

$$-\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{-s}e^{st}}{s+1}\, ds = -\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{s(t-1)}}{s+1}\, ds\,.$$

Case: $t < 1$. Use contour to the right to conclude that $x = 0$.

Case: $t > 1$. Simple pole at $s = -1$. Use contour to the left to conclude that

$$x = \text{Re}_{-1}(-\frac{e^{s(t-1)}}{s+1}) = -e^{-(t-1)}$$

The argument showing that the integral over C_R vanishes in the limit needs using the Lebsgue Theorem as well.

Summing up, we get the result coinciding with the elementary solution.

3. Elementary ODEs. Banach Contractive Map Theorem.
Consider an initial-value problem:

$$\begin{cases} q \in C([0,T]) \cap C^1(0,T) \\ \dot{q} = \frac{q^2}{t-1} \\ q(0) = 1 \end{cases}$$

(a) Solve the problem by elementary means.

(b) State (precisely) the Contractive Map Theorem.

(c) Use the theorem to determine a concrete value of T for which the problem has a unique solution. Compare with (a).

(20 points)

Solution:

(a) Use separation of variables:

$$\frac{dq}{q^2} = \frac{dt}{t-1} \qquad \Rightarrow \qquad -\frac{1}{q}|_1^{q(t)} = \ln|t-1||_0^t \,.$$

We obtain:

$$q(t) = \frac{1}{1 - \ln|t-1|} \,.$$

The solution has a blow up at $t = 1$.

(b) See the book.

(c) The problem is equivalent to the integral equation:

$$q(t) = 1 + \int_0^t \frac{q^2(s)}{s-1}\, ds \,.$$

Consider the Chebyshev space $C([0,T])$ where T is to be determined. We need to identify a subset $B \subset C([0,T])$ such that the nonlinear map

$$A : B \to B \quad (Aq)(t) = 1 + \int_0^t \frac{q^2(s)}{s-1}\, ds$$

is, first of all, well-defined. As $(Aq)(0) = 1$, it is natural to define the set as

$$B := \{q \in C([0,T]) \,:\, |q(t) - 1| \leq 1 \quad t \in [0,T]\} \,.$$

As a closed subset of a complete space, set B is complete as well. Constant 1 used in the bound is somehow arbitrary.

Step 1: We first determine T that guarantees that A maps set B into itself. First of all, if the integrand is bounded then Aq is Lipschitz continuous as,

$$|(Aq)(t_2) - (Aq)(t_1)| \le \int_{t_1}^{t_2} \underbrace{|\frac{q^2(s)}{s-1}|}_{\le C} \, ds \le C(t_2 - t_1)\,.$$

The very boundedness of the integrand indicates that the maximum T cannot exceed $t = 1$ at which the integrand blows up, i.e. $T < 1$.

We first estimate $q(t)$,

$$|q(t) - 1| \le 1 \quad \Rightarrow \quad -1 \le q(t) - 1 \le 1 \quad \Rightarrow \quad 0 \le q(t) \le 2 \quad \Rightarrow \quad q^2(t) \le 4\,.$$

We now determine under what conditions on T, $q \in B$. Using a rather conservative estimate,

$$|(Aq)(t) - 1| = |\int_0^t \frac{q^2(s)}{s-1}\, ds| \le 4\int_0^t \frac{1}{1-s}\, ds \le 4\frac{1}{1-T}\int_0^t ds \le \frac{4T}{1-T}\,,$$

we request,

$$\frac{4T}{1-T} \le 1 \quad \Rightarrow \quad T \le \frac{1}{5}\,.$$

Thus the map is well-defined provided $T \le \frac{1}{5}$.

Step 2: We check, under what condition on T, map A is contractive. We have,

$$\begin{aligned} |(Aq_1)(t) - (Aq_2)(t)| &\le \int_0^t \frac{|(q_1(s))^2 - (q_2(s))^2|}{|s-1|}\, ds \\ &= \int_0^t \frac{|(q_1(s) - q_2(s))(q_1(s) + q_2(s))|}{|s-1|}\, ds \\ &\le \max_{t\in[0,T]} |q_1(t) - q_2(t)| \int_0^t \frac{|(q_1(s) + q_2(s))|}{|s-1|}\, ds\,. \end{aligned}$$

We need to determine, under what conditions the factor on the right is strictly less than one. Following the same estimation as above, we get:

$$\int_0^t \frac{|(q_1(s) + q_2(s))|}{|s-1|}\, ds \le \frac{4T}{1-T} < 1 \quad \Rightarrow \quad T < \frac{1}{5}\,.$$

The map is thus a contraction if $T < \frac{1}{5}$. In conclusion, for any $T < \frac{1}{5}$, map $A : B \to B$ is a well-defined contraction and, by the Banach Contractive Map Theorem, the integral equation has a unique solution.

Comparing with solution from (a), we see that our estimate is quite conservative. The solution exists for $T < 1$, it blows up at $T = 1$.

4. Separation of variables. Solve the heat conduction problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \alpha^2 \dfrac{\partial^2 u}{\partial x^2} &= 0 & x \in (0,l),\ t > 0 \\ u(0,t) = u(l,t) &= 0 & t > 0 \\ u(x,0) &= 1 & x \in (0,l)\,. \end{cases}$$

where $\alpha, l > 0$. Explain the relation with the Sturm-Liouville Theorem and deliver the formula for the final solution. (20 points)

Solution: Separating the variables: $u = X(x)T(t)$, we obtain,

$$-\frac{1}{\alpha^2}\frac{T'}{T} = -\frac{X''}{X} = \lambda\,.$$

Self-adjointness in $L^2(0,l)$ and positive definitness of $Au = -u''$ with the homogenous Dirichlet BCs, implies that $\lambda = k^2$, $k > 0$. The Sturm-Liouville problem in x leads to the solutions:

$$X_n = \sin k_n x, \quad k_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots\,.$$

In turn, the ODE in t gives:

$$T_n = e^{-\alpha^2 k_n^2 t}\,.$$

By superposition, the ultimate solution is:

$$u = \sum_{n=1}^{\infty} c_n \sin k_n x e^{-\alpha^2 k_n^2 t}\,.$$

Constants c_n are determined from the IC:

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin k_n x = u_0(x) = 1, \quad x \in (0,l)\,.$$

It is critical that $\sin k_n x$ provide an orthogonal basis for $L^2(0,l)$. Multiplying both sides of the equation above with $\sin k_n x$, integrating over $(0,l)$, and using the L^2-orthogonality condition, leads to formulas for the coefficients c_n in terms of $u_0 = 1$,

$$c_n = \frac{2}{l}\int_0^l u_0(x) \sin k_n x\, dx\,.$$

More precisely,

$$c_n = \frac{2}{l}\int_0^1 \sin\frac{n\pi x}{l}\, dx = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

and the ultimate solution is of the form:

$$u(x,t) = \sum_{\text{odd } n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{l}x\right) e^{-\frac{n^2\pi^2\alpha^2}{l^2}t}\,.$$

5. Jordan Theorem and its relation with a system of linear ODEs.

Consider the matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

- Determine generalized eigenvectors of matrix A and the corresponding Jordan form.
- Use the Jordan form to determine general solution to the system of ODEs:

$$\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{u}\,.$$

(20 points)

Solution: The matrix is upper triangular, so the terms on the diagonal are the eigenvalues, we have a single eigenvalue $\lambda = 1$, and a double eigenvalue $\lambda = 2$. Solving for the eigenvector corresponding to $\lambda = 1$,

$$(\boldsymbol{A} - 1\boldsymbol{I})\boldsymbol{x} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = (t, 0, 0)^T\,.$$

We can choose $t = 1$, getting $\boldsymbol{e}_1 = (1, 0, 0)^T$. Solving for eigenvectors corresponding to $\lambda = 2$,

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = (t, t, 0)^T\,.$$

We have only one eigenvector. Solving for the corresponding generalized eigenvector in the chain:

$$(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{x} = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{x} = (u, u + \frac{1}{3}t, \frac{1}{3}t)^T\,.$$

We can choose $t = 1,\ u = 1$, getting $\boldsymbol{e}_2 = (1, 1, 0)^T, \boldsymbol{e}_3 = (1, \frac{4}{3}, \frac{1}{3})$.

By the Jordan Theorem, matrix $\boldsymbol{A}$ takes the following form in the eigenbasis.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus, seeking the solution to the system of ODEs in the eigenbasis,

$$\boldsymbol{x} = c_1(t)\boldsymbol{e}_1 + c_2(t)\boldsymbol{e}_2 + c_3(t)\boldsymbol{e}_3\,,$$

we obtain the following system of equations:

$$\begin{cases} \dot{c_1} &= c_1 \\ \dot{c_2} &= 2c_2 \\ \dot{c_3} &= c_2 + 2c_3 \end{cases}$$

This leads to: $c_1 = C_1 e^t$, $c_2 = C_2 e^{2t}$, $c_3 = (C_2 t + C_3) e^{2t}$ and the final formula for the solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (C_2 t + C_3) e^{2t} \begin{pmatrix} 1 \\ \frac{4}{3} \\ \frac{1}{3} \end{pmatrix}.$$