CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 31, 2018, 9:00 a.m. – 12:00 noon

Work any 5 of the following 6 problems.

1. The set $\mathcal{X}$ of all sequences $\{x_n\}_{n=1}^{\infty}$ of complex numbers is a vector space. Let $0 < p < 1$ and let $X \subset \mathcal{X}$ be the set of all sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

(a) Show that X is a vector space. [Hint: Show that $|x+y|^p \leq 2^{p-1}(|x|^p + |y|^p)$.]

(b) Show that the map taking $\{x_n\}_{n=1}^{\infty} \in X$ to $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is *not* a norm on X.

(c) Show that the map $d : X \times X \to \mathbb{R}$ defined by $d(\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} |x_n - y_n|^p$ is a metric on X.

2. Open Mapping Theorem.

(a) State the Open Mapping Theorem.

(b) Suppose that $\|\cdot\|$ and $\|\cdot\|'$ are two norms on a vector space X. Suppose that both $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are complete and there is a constant $C > 0$ such that

$$\|x\| \leq C\|x\|' \quad \text{for all } x \in X.$$

From the Open Mapping Theorem, show that the two norms are equivalent.

3. Let $\Omega = [a, b]$, $p, q \in (1, \infty)$, and $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in L^q(\Omega)$. For every $u \in L^p(\Omega)$ define a function Au by setting

$$(Au)(t) = \int_a^t v(s)\, u(s)\, ds \quad \text{for all } t \in \Omega.$$

(a) Show that A maps $L^p(\Omega)$ into $L^p(\Omega)$ and is continuous.

(b) Explain why $A : L^p(\Omega) \to L^p(\Omega)$ is compact.

4. Suppose (X, d_X) and (Y, d_Y) are metric spaces, Y is complete, $A \subset X$ is dense, and $T : A \to Y$ is uniformly continuous. Prove that there is a unique extension $\tilde{T} : X \to Y$ which is uniformly continuous.

5. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $f \in L^2(\Omega)$, and $\epsilon > 0$. Suppose u_ϵ satisfies

$$
\begin{aligned}
-\epsilon\Delta u_\epsilon + u_\epsilon = f &\quad \text{in } \Omega,\\
u_\epsilon = 0 &\quad \text{on } \partial\Omega.
\end{aligned}
$$

Show $u_\epsilon \to f$ in $L^2(\Omega)$ as $\epsilon \to 0$. [Hint: Bound appropriate norms of u_ϵ and $\sqrt{\epsilon}u_\epsilon$.]

6. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary and outer unit normal ν. Let $\boldsymbol{b}$ a constant vector and $f \in L^2(\Omega)$. Consider the fourth order problem

$$
\begin{aligned}
u + \Delta^2 u + \boldsymbol{b}\cdot\nabla u = f &\quad \text{in } \Omega,\\
u = 0 \quad \text{and} \quad \nabla u\cdot\nu = 0 &\quad \text{on } \partial\Omega.
\end{aligned}
$$

(a) State the Lax-Milgram Theorem for a real Hilbert space.

(b) Develop a suitable variational form for the problem. [Be careful to handle the boundary values and define the Hilbert spaces you use.]

(c) Give a hypothesis on $|\boldsymbol{b}|$ so that the Lax-Milgram theorem provides a unique solution to your variational problem. [Hint: Gårding's inequality gives a $C_G > 0$ such that $\|v\|_{H^2}^2 \le C_G\{\|u\|^2 + \|\Delta u\|^2\}$ for all $v \in H_0^2$.]

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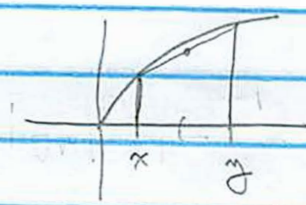
1. $0 < p < 1$, $\{x_n\}$, $\sum |x_n|^p < \infty$

(a) (i) $\{x_n\} + \{y_n\} = \{x_n + y_n\}$



$$f\left(\frac{x+y}{2}\right) < \frac{1}{2}\left(f(x) + f(y)\right)$$

$$\Rightarrow \frac{|x+y|^p}{2^p} < \frac{1}{2}\left(|x|^p + |y|^p\right)$$

$$\Rightarrow \sum |x_n + y_n|^p \leq 2^{p-1}\left(\sum x_n^p + \sum y_n^p\right) < \infty$$

(ii) $\alpha \{x_n\} = \{\alpha x_n\}$

$$\sum |\alpha x_n|^p = \alpha^p \sum |x_n|^p < \infty.$$

(b) Consider the Δ ineq. for

$x = (1, 0, 0, \ldots)$ and $y = (0, 1, 0, 0, \ldots)$

$$\Rightarrow \left(\sum |(x+y)_n|^p\right)^{1/p} = 2^{1/p} > 2$$

$$\overset{?}{\leq} \left(\sum |x_n|^p\right)^{1/p} + \left(\sum |y_n|^p\right)^{1/p} = 1 + 1 = 2$$

But $2^{1/p} > 2$, so not a norm.

(c) $d(x,y) = \sum |x_n - y_n|^p$

(i) $d(x,y) \geq 0$ ✓, $d(x,y) = 0 \iff x_n = y_n \ \forall n$ ✓.

(ii) $d(x,y) = d(y,x)$ ✓

(iii) $d(x,y) \overset{?}{\leq} d(x,z) + d(z,y)$

$$|x_n - y_n|^p = |(x_n - z_n) + (z_n - y_n)|^p$$

$$\leq 2^{p-1}\left(|x_n - z_n|^p + |z_n - y_n|^p\right) \quad (\text{see (a)})$$

$$\leq |x_n - z_n|^p + |z_n - y_n|^p \quad (p - 1 < 0).$$

$$\Rightarrow \sum |x_n - y_n|^p \leq \sum |x_n - z_n|^p + \sum |z_n - y_n|^p. \ ✓$$

2. Open Mapping

(a) Let X, Y be Banach

If $T: X \to Y$ is bounded, linear, surjective

Then T is open (maps open sets to open sets).

(b) $\|\cdot\|, \|\cdot\|'$, $(X, \|\cdot\|)$ & $(X, \|\cdot\|')$ complete.

$\|x\| \leq C \|x\|' \quad \forall x \in X$.

Consider

$$i : (X, \|\cdot\|') \to (X, \|\cdot\|) \qquad \text{identity}$$

Note i is bounded, surjective (and linear),

so i is open $\Rightarrow$

$$i : B_1' \xrightarrow{\text{onto}} Q \subseteq X \text{ open}$$

Now $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon \subseteq Q$

Given $x \in X$,

$$\frac{\varepsilon}{2} \frac{x}{\|x\|} \in B_\varepsilon \cap B_1'$$

$$\Rightarrow \quad \frac{\varepsilon}{2} \left\| \frac{x}{\|x\|} \right\|' \leq 1 \Rightarrow \|x\|' \leq \frac{2}{\varepsilon} \|x\|.$$

Thus the norms are equivalent.

3. $\Omega = [a,b]$, $p, q \in (1,\infty)$, $\frac{1}{p}+\frac{1}{q}=1$.
$v \in L^q(\Omega)$. $A : L^p(\Omega) \longrightarrow$ fcns.

$$(Au)(t) = \int_a^t v(s)\,u(s)\,ds \qquad \forall t \in \Omega$$

(a) $A : \longrightarrow L^p(\Omega)$

$$\int_a^b |(Au)(t)|^p = \int_a^b \left| \int_a^t v(s)u(s)\,ds \right|^p dt$$
$$\leq \int_a^b \left(\|v\|_{L^q} \|u\|_{L^p} \right)^p dt$$
$$\leq (b-a)\, \|v\|_{L^q}^p \|u\|_{L^p}^p$$

$\Rightarrow$

$$\|Au\|_{L^p} \leq (b-a)^{1/p} \|v\|_{L^q} \|u\|_{L^p}$$

So A maps into L^p and is continuous.

(b)
$$Au(t) = \int_a^b \underbrace{v(s)\chi_{[a,t]}(s)}_{\in L^q(\Omega,\Omega)}\, u(s)\,ds$$

Use density of $C(\Omega)$ in L^p, L^q

$v_j \xrightarrow{L^q} v$, $u_k \xrightarrow{L^p} u$

$A_j u = \lim_k A_j u_k$, $A_j : C(\Omega) \to C(\Omega)$ is compact by Ascoli-Arzela

$A_j : L^p \to L^p$ compact by density.
$A_j \to A$ is also compact.

4. X, Y metric, Y complete, $A \subseteq X$ dense
$T: A \to Y$ unif. cont. (on A).

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0$ s.t. $\forall x, y \in A$,
$d_Y(Tx, Ty) < \varepsilon$ whenever $d_X(x,y) < \delta_\varepsilon$

Let $x \in X$ and $x_n \to x$, $x_n \in A$
Claim: $\{Tx_n\}$ Cauchy.
$\{x_n\}$ Cauchy $\Rightarrow \exists N > 0$ s.t. $d_X(x_n, x_m) < \delta_\varepsilon$
$\forall n, m > N \Rightarrow$
$d_Y(Tx_n, Tx_m) < \varepsilon \quad \forall n, m > N$.
Y complete $\Rightarrow Tx_n \to z \equiv \tilde{T}(x)$.

Claim: $\tilde{T}$ unif. cont. If so, then $\tilde{T}x = Tx \ \forall x \in A$
and $\tilde{T}$ is unique (since A dense).
Now $d_Y(\tilde{T}x, \tilde{T}y) \leq d_Y(\tilde{T}x, Tx_n) + d_Y(Tx_n, Ty_m) + d_Y(Ty_m, \tilde{T}y)$
where $x_n \to x$, $y_m \to y$, $x_n, y_m \in A$.
If $N > 0$ chosen so
$$d_X(x_n, y_m) < d_X(x_n, x) + d_X(x, y) + d_X(y, y_m)$$
$$< \delta_\varepsilon/3 + \delta_\varepsilon/3 + \delta_\varepsilon/3$$
for $n, m > N$ and $d_X(x,y) < \delta_\varepsilon/3$.
$\Rightarrow \quad d_Y(\tilde{T}x, \tilde{T}y) \leq 3\varepsilon$ for n, m large.

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5. $\Omega \subseteq \mathbb{R}^d$, $f \in L^2$, $\varepsilon > 0$

$-\varepsilon \Delta u_\varepsilon + u_\varepsilon = f$, Ω; $u_\varepsilon = 0$, $\partial\Omega$

Equiv. variational form is.

$$\varepsilon(\nabla u_\varepsilon, \nabla v) + (u_\varepsilon, v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

$v = u_\varepsilon \Rightarrow$

$$\varepsilon \|\nabla u_\varepsilon\|^2 + \|u_\varepsilon\|^2 = (f, u_\varepsilon) \leq \|f\| \, \|u_\varepsilon\|$$
$$\leq \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_\varepsilon\|^2$$

$\Rightarrow \quad \varepsilon \|\nabla u_\varepsilon\|^2 + \tfrac{1}{2}\|u_\varepsilon\|^2 \leq \tfrac{1}{2}\|f\|^2$

$\Rightarrow \quad \sqrt{\varepsilon}\, u_\varepsilon$ bounded in H_0^1

u_ε " " L^2

$\Rightarrow \quad u_\varepsilon \rightharpoonup u$ in L^2

$\sqrt{\varepsilon}\, u_\varepsilon \rightharpoonup g$ in $H_0^1 \Rightarrow \sqrt{\varepsilon}\, u_\varepsilon \rightarrow g$ in L^2

But $\underbrace{\sqrt{\varepsilon}}_{\downarrow 0}\, \underbrace{u_\varepsilon}_{\rightharpoonup u} \rightarrow 0 \Rightarrow g = 0$

Thus

$$0 + (u, v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

$\Rightarrow \quad (u - f, v) = 0 \Rightarrow u = f$

6. $\Omega \subseteq \mathbb{R}^d$, $\underline{b}, f \in L^2$

$$\begin{cases} u + \Delta^2 u + b \cdot \nabla u = f, & \Omega \\ u = 0, \quad \nabla u \cdot \nu = 0, & \partial\Omega \end{cases}$$

(a) Let $\mathcal{H}$ be a real Hilbert space with closed subspace H. Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be bilinear s.t.

(i) $|B(x,y)| \leq M \|x\| \, \|y\| \quad \forall x, y \in \mathcal{H}$. (cont)

(ii) $B(x,x) \geq \gamma \|x\| \quad \forall x \in H$ (coercive)

If $x_0 \in \mathcal{H}$, $F \in H^*$, then $\exists !\, u \in H + x_0$ s.t.

$$B(u,v) = F(v) \quad \forall v \in H.$$

$$\|u\| \leq \frac{1}{\gamma}\|F\| + \left(\frac{M}{\gamma} + 1\right)\|x_0\|$$

(b) Find $u \in H = \{w \in H^2 : w = 0, \nabla w \cdot \nu = 0 \text{ on } \partial\Omega\}$ s.t.

$$(u,v) + (\Delta u, \Delta v) + (b \cdot \nabla u, v) = (f, v) \quad \forall v \in H.$$

(c) $LHS = B(u,v)$, which is cont. For coercivity:

$$\|u\|^2 + \|\Delta u\|^2 + (b \cdot \nabla u, u)$$
$$\geq \|u\| + \|\Delta u\|^2 - |b| \|\nabla u\| \|u\|$$
$$\geq \|u\| + \|\Delta u\|^2 - \frac{4}{\varepsilon}|b|^2 \|\nabla u\|^2 - \varepsilon \|u\|^2$$
$$= (1-\varepsilon)\|u\|^2 + \|\Delta u\|^2 - \frac{4}{\varepsilon}|b|^2 \|\nabla u\|^2$$

Now $C_G(\|u\|^2 + \|\Delta u\|^2) \geq \|\nabla u\|^2 \Rightarrow$ Need

$$(1-\varepsilon)\frac{4}{\varepsilon}|b| < C \Rightarrow \boxed{|b| < \frac{C_G}{4}}$$

CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 30, 2019, 9:00 a.m. – 12:00 noon

Work any 5 of the following 6 problems.

1. Let X be a Banach space with dual space X^* and duality pairing $\langle \cdot, \cdot \rangle$, and let $A, B : X \to X^*$ be linear maps.

(a) State the Closed Graph Theorem and what it means for an operator to be closed.

(b) Assuming $\langle Ax, y \rangle = \langle Ay, x \rangle$ for all $x, y \in X$, show that A is bounded.

(c) Assuming $\langle Bx, x \rangle \geq 0$ for all $x \in X$, show that B is bounded. [Hint: Suppose B is not continuous at 0, so $x_n \to 0$ but $Bx_n \to y \neq 0$. For $w \in X$ such that $\langle y, w \rangle > 0$, consider $x_n + \epsilon w$.]

2. Let $\Omega = [0, 1]$ and $1 \leq p < \infty$ be given and consider the sequence of functions $g_n \in L^p(\Omega)$ defined by $g_n(x) = n^{1/p} e^{-nx}$. Show that as $n \to \infty$:

(a) $g_n(x)$ converges pointwise to zero for each fixed $x \in (0, 1]$ and for any $p \geq 1$;

(b) g_n does not converge strongly to zero in $L^p(\Omega)$ for any $p \geq 1$;

(c) g_n converges weakly to zero in $L^p(\Omega)$ if $p > 1$, but not if $p = 1$.

3. Prove the Mazur Separation Lemma, which says that if X is a normed linear space, Y a linear subspace of X, $w \in X$ but $w \notin Y$, and

$$d = \operatorname{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0,$$

then there exists $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(z) = 0$ for all $z \in Y$. [Hint: Begin by working in $Z = Y + \mathbb{F}w$.]

4. Let $\Omega = (0, 1)^2$ and consider the boundary value problem (BVP)

$$-u_{xx} + u_{xy} - u_{yy} = f \quad \text{in } \Omega, \tag{1}$$

$$-u_x + u_y - u = g \quad \text{on } \Gamma_L = \{(0, y) : y \in (0, 1)\}, \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_* = \partial\Omega \setminus \Gamma_L. \tag{3}$$

Let $H = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_*\}$, which is a Hilbert space.

(a) Find the corresponding variational problem for $u \in H$ and test functions $v \in H$. Also give the function spaces containing f and g.

(b) Show the general Poincaré type inequality: There exists $\gamma > 0$ such that

$$\|\nabla v\|^2_{L^2(\Omega)} + \int_{\Gamma_L} v^2 \geq \gamma \|v\|^2_{L^2(\Omega)} \quad \forall v \in H.$$

(c) Show that there is a unique solution to the variational problem.

5. For fixed $T > 0$, let $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and Lipschitz continuous in the second argument, i.e., there is some $L > 0$ such that

$$\|g(t, v) - g(t, w)\| \leq L \|v - w\| \quad \forall v, w \in \mathbb{R}^d, t \in [0, T],$$

where $\|\cdot\|$ is the norm on $\mathbb{R}^d$. For any $u_0 \in \mathbb{R}^d$, consider the initial value problem (IVP) $u'(t) = g(t, u(t))$ and $u(0) = u_0$.

(a) Write this IVP as the fixed point of a functional $G : C^0([0, T]; \mathbb{R}^d) \to C^0([0, T]; \mathbb{R}^d)$.

(b) Normally, we use the $L^\infty([0, T])$-norm for $C^0([0, T]; \mathbb{R}^d)$. Show that the function $|||\cdot||| : C^0([0, T]; \mathbb{R}^d) \to [0, \infty)$, defined by

$$|||v||| = \sup_{0 \leq t \leq T} \left(e^{-Lt} \|v(t)\| \right),$$

is a norm equivalent to the $L^\infty([0, T])$-norm.

(c) In terms of this new norm, show that G is a contraction.

(d) Explain how we conclude that there is a unique solution $u \in C^1([0, \infty); \mathbb{R}^d)$ to the IVP for all time.

6. Consider finding extremals to the problem: Find $u, v \in C^1_{0,1}([0, 1])$ minimizing

$$F(u, v, u', v') = \int_0^1 \left((u')^2 + (v')^2 + 2uv \right) dx.$$

(a) Find the Euler-Lagrange (EL) equations for this problem.

(b) Reduce the EL equations to a single equation and find its solution. [Hint: The fourth roots of unity are ± 1 and $\pm i$.]

(c) Find the extremal to the problem, up to solving a 4×4 system of linear equations.

(d) If we add the constraint that $\displaystyle\int_0^1 u^2 v' \, dx = 0$, what EL equations do we get?

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Solutions

1. X Banach, $A, B: X \to X^*$ linear.

(a) Closed Graph Theorem:
Let X and Y be Banach spaces and $T: X \to Y$ linear. Then: T is continuous (bnded)
$\iff$ T is closed

T is closed if whenever $x_n \xrightarrow{X} x$, $Tx_n \xrightarrow{Y} y$, then $y = Tx$.

(b) $\langle Ax, y \rangle = \langle Ay, x \rangle \quad \forall x, y \in X$

Suppose $x_n \xrightarrow{X} x$, $A x_n \xrightarrow{X^*} y$

Then $\langle A x_n, z \rangle = \langle Az, x_n \rangle \quad \forall z \in X$

$\to \quad \langle y, z \rangle = \langle Az, x \rangle = \langle Ax, z \rangle$

$\Rightarrow \quad Ax = y$, and A continuous (bnded).

(c) $\langle Bx, x \rangle \geq 0 \quad \forall x \in X$

ETS for $x_n \to 0$, $Bx_n \to y \overset{?}{=} 0$

Suppose not: $y \neq 0$ so $\exists\, w \in X$, $\langle y, w \rangle \neq 0$

Consider

$$0 \leq \langle B(x_n + \varepsilon w), x_n + \varepsilon w \rangle$$
$$\to \langle y + \varepsilon Bw, \varepsilon w \rangle$$
$$= \varepsilon \langle y, w \rangle + \underbrace{\varepsilon^2 \langle Bw, w \rangle}_{\geq 0}$$

Let $\varepsilon \to 0$ so last term negligible.
Contradiction, since ε can be $+$ or $-$.
So $y = 0$ and B cont.

2. $\Omega = [0,1]$, $1 \le p < \infty$, $g_n(x) = n^{1/p} e^{-nx}$.

(a) $g_n(x) = \dfrac{n^{1/p}}{e^{nx}} \xrightarrow[\text{L'Hôpital}]{} \dfrac{\frac{1}{p} n^{1/p - 1}}{x e^{nx}} \longrightarrow 0.$

(b) $\|g_n\|_{L^p}^p = \int_0^1 n e^{-npx}\, dx = -\frac{1}{p} e^{-npx}\Big|_0^1$

$= \frac{1}{p}(1 - e^{-np}) \longrightarrow 1/p \neq 0.$

So $g_n \not\to 0$.

(c) Let $h \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

If $p > 1$, then (by density) suppose $h \in C_0^1$.

Then $\exists\ x_* > 0$ st. $h(x) \equiv 0$ for $x < x_*$.

Now $\left| \int_0^1 g_n h \right| \le \int_0^1 g_n |h|$,

so suppose $h \ge 0$.

Note $\dfrac{d}{dn}\left(n^{1/p} e^{-nx} h \right) = n^{1/p} e^{-nx} \left(\dfrac{1}{pn} - x \right) h$

≤ 0 for n large enough, and $x \ge x_*$ (so $\forall x$).

Thus $g_n h$ is monotone, so MCT $\Rightarrow$

$\lim \int g_n h = \int \lim g_n h = 0.$

That is, $g_n \rightharpoonup 0$.

But for $p = 1$, $(L^1)^* = L^\infty$. Consider $h \equiv 1$.

Then $\int_0^1 n e^{-nx} = -e^{-nx}\Big|_0^1 = 1 - e^{-n} \longrightarrow 1 \neq 0.$

3. X NLS, Y lin. subsp., $w \in X \setminus Y$.

$$d = \mathrm{dist}(w, Y) = \inf_{y \in Y} \|w - y\| > 0.$$

Work in $Z = Y + \mathbb{F} w$

$z \in Z \Rightarrow \exists!\ y \in Y,\ \lambda \in \mathbb{F}$ s.t.

$$z = y + \lambda w$$

(for otherwise $Y \ni y - y' = (\lambda - \lambda') w \notin Y$).

Let $g : Z \to \mathbb{F}$

$g(z) = \lambda d$ (well defined).

Now g linear and

$$\frac{|g(y+\lambda w)|}{\|y + \lambda w\|} = \frac{|\lambda| d}{\|y + \lambda w\|} = \inf_{z \in Y} \frac{|\lambda| \|w - z\|}{\|y + \lambda w\|}$$

$$= \inf_{z \in Y} \frac{\|-\lambda z + \lambda w\|}{\|y + \lambda w\|} \leq 1.$$

$\Rightarrow \|g\| \leq 1.$

Extend (using Hahn-Banach) to X.

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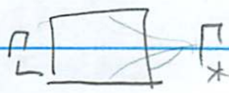
4. $\Omega = (0,1)^2$ $\begin{cases} -u_{xx} + u_{xy} - u_{yy} = f, & \Omega \\ -u_x + u_y + u = g, & \Gamma_L \\ u = 0, & \Gamma_* = \partial\Omega \setminus \Gamma_L \end{cases}$



$H = \{ v \in H^1 : v = 0 \text{ on } \Gamma_* \}$

(a) $(u_x, v_x) - \langle u_x, v \rangle_{\Gamma_L}$

$- (u_y, v_x) + \langle u_y, v \rangle_{\Gamma_L}$

$+ (u_y, v_y) \qquad\qquad = (f, v)$

$\Rightarrow$

$B(u,v) = (u_x, v_x) - (u_y, v_x) + (u_y, v_y) + \langle u, v \rangle$

$= (f, v) - \langle g, v \rangle_{\Gamma_L}$

So $f \in H^*$, $g \in (H^{1/2}(\Gamma_L))^*$

(b) Suppose not, so $\exists\ v_n$ s.t. $\|v_n\|_{L^2} = 1$

but $\|\nabla v_n\|_{L^2}^2 + \int_{\Gamma_L} v_n^2 \leq n.$

$\Rightarrow$ (subseq.)

$\nabla v_n \longrightarrow 0, \quad \int_{\Gamma_L} v_n^2 \longrightarrow 0$

$\|v_n\|_{H^1} \leq 2 \Rightarrow v_n \xrightarrow{H^1} v, \quad v_n \xrightarrow{L^2} v$

But $\nabla v = 0 \Rightarrow v = \text{const} \Rightarrow v = 0,$

which contradicts $\|v_n\|_{L^2} = 1.$

(c) Lax-Milgram. Linear form good by f, g.

Continuity: $|B(u,v)| \leq (\|u_x\| + \|u_y\|)(\|v_x\| + \|v_y\|) + \|u\|\,\|v\|$

$\leq \|u\|_{H^1} \|v\|_{H^1}$

Coercivity: $B(v,v) \geq \|v_x\|^2 - \underbrace{|(v_y, v_x)|}_{\geq -\frac{1}{2}\|v_y\|^2 + \|v_x\|^2} + \|v_y\|^2 + \int_{\Gamma_L} v^2$

$\geq \frac{1}{2}\gamma \|v\|_{H^1}$ by (b).

5. $u' = g(t, u(t)), \quad u(0) = u_0.$

(a) $u(t) - u(0) = \int_0^t g(s, u(s))\, ds$

$\Rightarrow$

$$G(u) = u_0 + \int_0^t g(s, u(s))\, ds$$

(b) $|||v||| = \sup_{0 \le t \le T} \left(e^{-Lt} \|v(t)\| \right)$

Note: $|||v||| \le \|v\|_{L^\infty}, \quad |||v||| \ge e^{-LT} \|v\|_{L^\infty}$

so $|||\cdot|||$ equiv. to $\|\cdot\|_{L^\infty} \Rightarrow$
$|||\cdot|||$ satisfies the zero property
Scaling clearly okay

$|||v+w||| = \sup_t \left(e^{-Lt} \|v+w\|\right) \le \sup_t e^{-Lt}(\|v\|+\|w\|)$
$\le \sup_t e^{-Lt}\|v\| + \sup_t e^{-Lt}\|w\|$
$= |||v||| + |||w|||$

so $|||\cdot|||$ is a norm.

(c) $e^{-Lt} \|G(v) - G(w)\| = e^{-Lt} \left\| \int_0^t (g(s,v) - g(s,w)) \right\|$
$\le L e^{-Lt} \int_0^t \|v-w\| = L e^{-Lt} \int_0^t e^{Ls} e^{-Ls} \|v-w\|$
$\le L e^{-Lt} \int_0^t e^{Ls} \; |||v-w|||$
$= e^{-Lt} \; e^{Ls}\Big|_0^t \; |||v-w||| = (1 - e^{-Lt}) |||v-w|||.$

$\Rightarrow \; |||G(v) - G(w)||| \le \theta |||v-w|||, \quad \theta = 1 - e^{-Lt} < 1.$

(d) Banach contraction mapping Thm $\Rightarrow \exists!$
$u \in C^0([0,T])$ s.t. $G(u) = u$ (i.e., IVP).
But if $u \in C^0$, then $G(u) \in C^1$
$\Rightarrow u \in C^1.$
Finally, let $T \to \infty.$

6. $u, v \in C'_{0,1}([0,1])$, $F(u,v,u',v') = \int_0^1 \underbrace{[(u')^2 + (v')^2 + 2uv]}_{f} dx$

(a) $f_{y_i} = (f_{y_i'})'$, $i = 1,2$

$$\begin{cases} 2v = 2u'' \\ 2u = 2v'' \end{cases} \Rightarrow \begin{cases} v = u'' \\ u = v'' \end{cases}$$

(b) $u = v'' = u''''$

$\Rightarrow \quad u = e^{rt}, \; r^4 = 1 \quad (r = \pm 1, \pm i)$

$u(x) = Ae^x + Be^{-x} + Ce^{ix} + De^{-ix}$

(c) $v(x) = u''$

$= Ae^x + Be^{-x} - Ce^{ix} - De^{-ix}$

$$\begin{cases} u(0) = A + B + C + D = 0 \\ u(1) = Ae + Be^{-1} + Ce^{i} + De^{-i} = 1 \\ v(0) = A + B - C - D = 0 \\ v(1) = Ae + Be^{-1} - Ce^{i} - De^{-i} = 1 \end{cases}$$

(d) $H = \int_0^1 [(u')^2 + (v')^2 + 2uv + \lambda u^2 v'] dx$

$$\begin{cases} 2v + 2\lambda uv' = 2u'' \\ 2u = 2v'' + \lambda(u^2)' \end{cases} \Rightarrow \begin{cases} v + \lambda uv' = u'' \\ u - \frac{1}{2}\lambda(u^2)' = v'' \end{cases}$$

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CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

August 7, 2020, about any 3 hours from 9:00 a.m. – 3:00 p.m.

You may use the class textbooks and your own notes on this exam.

Work any 5 of the following 6 problems.

1. A problem on continuous operators.

(a) Define the topological dual of a Banach space.

(b) Define the weak topology on a Banach space.

(c) Let X, Y be Banach spaces and $A : X \to Y$ be a linear operator. Prove that A is continuous if an only if it is weakly continuous (i.e., it is continuous when X and Y are equipped with their weak topologies).

Solution.

(a) The topological dual X' of a normed space X consists of all linear and continuous functionals defined on X. For a complex space X, we may define the topological dual as the space of all *anti*-linear and continuous functionals on X. Either space is equipped with the norm

$$l \in X', \quad \|l\|_{X'} := \sup_{x \in X, x \neq 0} \frac{|l(x)|}{\|x\|_X} = \sup_{\|x\|_X \leq 1} |l(x)| = \sup_{\|x\|_X = 1} |l(x)| \,.$$

For a reflexive Banach space, the supremum is actually attained and can be replaced with maximum. The dual space is always complete, no matter whether X is complete or not.

(b) The weak topology on a Banach space X is a locally convex topology defined by a family of seminorms

$$X \ni x \mapsto |\langle x', x \rangle| = |x'(x)|, \quad x' \in X'.$$

Due to the definitness of the duality pairing (proved using Hahn-Banach Theorem), the family of seminorms satisfies the axiom of separation which implies that the weak topology is well-defined.

(c) We first prove that weak continuity of A implies strong continuity of A. Assume, to the contrary, that there exists a sequence x_n such that $\|x_n\|_X \to 0$ but $\|Ax_n\|_Y \not\to 0$. At the cost of replacing x_n with a subsequence, we can assume that there exists $\epsilon > 0$ such that $\|Ax_n\|_Y \geq \epsilon$. Define,

$$\bar{x}_n = \frac{x_n}{\|x_n\|_X^{1/2}} \,.$$

Then,

$$\|\bar{x}_n\|_X = \|x_n\|_X^{1/2} \to 0 \quad \text{and} \quad \|A\bar{x}_n\|_Y \to \infty \,.$$

As the strong convergence implies weak convergence, $\bar{x}_n \rightharpoonup 0$ and, by weak continuity of A, $A\bar{x}_n \rightharpoonup 0$ in Y. But every weakly convergent sequence must be bounded, a contradiction.

Assume now that A is strongly continuous.

Lemma: Let X be an arbitrary topological vector space, and Y be a normed space. Let $A \in \mathcal{L}(X, Y)$. The following conditions are equivalent to each other.

(i) $A : X \to Y$ (with weak topology) is continuous.
(ii) $f \circ A : X \to \mathbb{R}(\mathbb{C})$ is continuous $\forall f \in Y'$.

(i) $\Rightarrow$ (ii). Any linear functional $f \in Y'$ is also continuous on Y with weak topology. Composition of two continuous functions is continuous.

(ii) $\Rightarrow$ (i). Take an arbitrary $B(I_0, \epsilon)$, where I_0 is a finite subset of Y'. By (ii),

$$\forall g \in I_0 \ \exists B_g, \text{ a neighborhood of } \mathbf{0} \text{ in } X \ : \ x \in B_g \ \Rightarrow \ |g(A(x))| < \epsilon \, .$$

It follows from the definition of filter of neighborhoods that

$$B = \bigcap_{g \in I_0} B_g$$

is also a neighborhood of $\mathbf{0}$. Consequently,

$$x \in B \ \Rightarrow \ |g(A(x))| < \epsilon \ \Rightarrow \ Ax \in B(I_0, \epsilon) \, .$$

To conclude the final result, it is sufficient now to show that, for any $g \in Y'$,

$$g \circ T \ : \ X \ (\text{with weak topology}) \ \to \mathbb{R}$$

is continuous. But $g \circ T$, as a composition of continuous functions, is a strongly continuous linear functional and, consequently, it is continuous in the weak topology as well (compare the discussion in the book).

2. Projections on a Hilbert space. Let X and Y be Hilbert spaces, $P : X \to Y$ and $Q : Y \to X$ be bounded linear operators, and suppose that $QP : X \to X$ is an orthogonal projection operator. Let $U_1 = R(QP)$ and $U_2 = N(QP)$, i.e., the image (or range) and null space (or kernel) of the operator, respectively. Moreover, let $V_1 = R(P)$.

(a) What does it mean to say $X = U_1 \oplus U_2$? Show that U_1 and U_2 are orthogonal to each other.

(b) Prove that U_1 and V_1 are isomorphic.

(c) Show directly that $P^*Q^* : X \to X$ is an orthogonal projection.

(d) If $N(Q) \cap R(PQ) = \{0\}$, show that $PQ : Y \to Y$ is a projection operator (not necessarily orthogonal).

Solution.

(a) The symbols $X = U_1 \oplus U_2$ mean that $X = \{u_1 + u_2 : u_i \in U_i, i = 1, 2\}$ and $U_1 \cap U_2 = \{0\}$. For $u_i \in U_i$, we know that $u_1 = QPu_1$ and $QPu_2 = 0$, so

$$\langle u_1, u_2 \rangle_X = \langle QPu_1, u_2 - QPu_2 \rangle_X = 0$$

by the definition of orthogonal projection.

(b) Consider the map $T = P|_{U_1} : U_1 \to V_1$, that is bounded and linear. Every $v \in V_1$ has some $u \in X$ such that $Pu = v$. However, there are (unique) $u_i \in U_i$ such that $u = u_1 + u_2$, and so $Tu_1 = Pu_1 = Pu = v$ shows that T maps onto V_1. To finish, we need to show that T maps one-to-one, i.e., that $Tu_1 = 0$ implies that $u_1 = 0$. But $0 = Tu_1 = Pu_1$, so also $QPu_1 = 0$. Thus $u_1 \in U_1 \cap U_2$, and so $u_1 = 0$.

(c) For $u, w \in X$, we compute

$$0 = \langle QPu - u, w \rangle_X = \langle u, P^*Q^*w - w \rangle_X,$$

which shows that P^*Q^* is also an orthogonal projection operator.

(d) For $y \in Y$, we know that $QPQPQy = QPQy$, since QP is a projection. But then

$$0 = QPQPQy - QPQy = Q(PQPQy - PQy) = QP(QPQy - Qy)\,.$$

Thus $PQPQy - PQy \in N(Q)$ and clearly $PQPQy - PQy \in R(PQ)$, so $PQPQy = PQy$.

3. Hilbert basis. Let H be a separable Hilbert space and let $\{e_n\}_{n=1}^{\infty}$ be a maximal orthonormal set (i.e., a Hilbert basis). Let $\{\lambda_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and define the linear operator $A : H \to H$ by

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n\,.$$

(a) Show that A is continuous and self-adjoint.

(b) Show that each λ_n is an eigenvalue with eigenvector e_n.

(c) Show that if $\lambda_n \to 0$, then A is compact. [Hint: Consider the operator A_N defined by a truncated sum, and show that A_N converges to A.]

Solution.

(a) If $x_m \to 0$, then $\|x_m\|^2 = \sum_{n=1}^{\infty} |\langle x_m, e_n \rangle|^2 \to 0$. Thus

$$\|Ax_m\| = \sum_{n=1}^{\infty} |\lambda_n|^2\, |\langle x, e_n \rangle|^2 \le \max_n |\lambda_n|^2 \sum_{n=1}^{\infty} |\langle x_m, e_n \rangle|^2 \to 0\,.$$

That is, A is continuous at 0, and so continuous everywhere.

Now

$$\langle Ax, y \rangle = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle \overline{\langle y, e_n \rangle} = \sum_{n=1}^{\infty} \langle x, e_n \rangle \overline{\lambda_n \langle y, e_n \rangle} = \langle x, Ay \rangle$$

is clearly self adjoint (since λ_n is real).

(b) Compute

$$(A-\lambda I)x = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n\rangle e_n - \lambda \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n = \sum_{n=1}^{\infty} (\lambda_n - \lambda)\langle x, e_n\rangle e_n\,,$$

and note that this cannot be invertible when $\lambda = \lambda_n$ for some n. Moreover, $Ae_n = \lambda_n e_n$ is clear by orthonormality of the basis.

(c) Consider the operators

$$A_N x = \sum_{n=1}^{N} \lambda_n \langle x, e_n\rangle e_n\,.$$

Each has finite dimensional range, and is hence compact. Moreover,

$$\|A_N x - Ax\|^2 = \Bigg\| \sum_{n=N+1}^{\infty} \lambda_n \langle x, e_n\rangle e_n \Bigg\|^2 = \sum_{n=N+1}^{\infty} |\lambda_n|^2 \, |\langle x, e_n\rangle|^2 \to 0\,,$$

so $A_n \to A$ and A is compact.

4. Closed operators. All spaces are real. Consider the operator

$$A : D(A) \to L^2(0,1), \quad Au = u' + u,$$
$$D(A) := \{u \in L^2(0,1) : Au \in L^2(0,1),\ u(0) = 0,\ u(1) = 0\},$$

where the derivative is understood in the sense of distributions.

(a) Interpret $D(A)$ in terms of Sobolev spaces.

(b) Show that A is a closed operator.

(c) Prove that A is bounded below in $L^2(0,1)$.

(d) Compute the L^2-adjoint A^*, $L^2(0,1) \supset D(A^*) \ni v \mapsto A^*v \in L^2(0,1)$.

(e) Compute the null space of the adjoint operator A^*.

(f) For an appropriate right-hand side f, discuss the well-posedness of the problem:

$$\begin{cases} u \in D(A), \\ Au = f\,. \end{cases}$$

Solution.

(a) We have

$$u, u' + u \in L^2(0,1) \quad \Leftrightarrow \quad u, u' \in L^2(0,1) \quad \Leftrightarrow \quad u \in H^1(0,1)\,.$$

Consequently, $D(A) = H^1_0(0,1)$.

(b) We need to show that

$$D(A) \ni u_n \to u, \quad Au_n \to w \quad \Rightarrow \quad u \in D(A),\ Au = w\,.$$

All convergence is in the L^2-sense. Let $\phi \in \mathcal{D}(0,1)$. We have

$$\begin{array}{ccccl} (u_n, -\phi') & + & (u_n, \phi) & = & (-u_n' + u_n, \phi) \to (w, \phi) \\ \downarrow & & \downarrow & & \\ (u, -\phi') & & (u, \phi) & & \end{array}$$

This proves that $-u' + u = w$ and, therefore, $u \in H^1(0,1)$. Moreover, $u_n \to u$ in $H^1(0,1)$. Continuous embedding of $H^1(0,1)$ into $C([0,1])$ implies that,

$$u(x) = \lim_{n\to\infty} u_n(x) = 0 \quad \text{for } x = 0, 1\,.$$

Consequently, $u \in D(A)$.

(c) We have

$$\|Au\|^2 = \|u'\|^2 + \|u\|^2 + 2(u', u)\,.$$

But

$$2(u', u) = \int_0^1 \frac{d}{dx}(u^2) = u^2|_0^1 = 0\,.$$

Consequently,

$$\|Au\|^2 = \|u'\|^2 + \|u\|^2 \geq \|u\|^2\,.$$

(d) Integration by parts and BC's on u reveal that

$$D(A^*) = H^1(0,1) \quad A^*v = -v' + v\,.$$

(e) We get

$$D(A^*) = \{ce^x \ :\ c \in \mathbb{R}\}\,.$$

(f) According to the Closed Range Theorem for Closed Operators, the equation has a unique solution u for every $f \in L^2(0,1)$ such that $f \in \mathcal{N}(A^*)^\perp$, i.e.,

$$\int_0^1 f(x)e^x = 0\,.$$

5. Variational formulations. Consider the *ultraweak* variational formulation of the previous problem, i.e.,

$$\left\{ \begin{array}{l} u \in L^2(0,1) =: U \\ \underbrace{\int_0^1 uA^*v\,dx}_{b(u,v)} = \underbrace{\int_0^1 fv\,dx}_{l(v)} \quad \forall v \in D(A^*) = H^1(0,1) =: V\,, \end{array} \right.$$

where A^* denotes the L^2-adjoint of A, $A^*v = -v' + v$, and $f \in L^2(0,1)$. [Hint: For this problem, use results of the previous problem.]

(a) Define the operator $B : U \to V'$ and its conjugate corresponding to the bilinear form $b(u,v)$.

(b) State the Babuška-Nečas Theorem for Hilbert spaces.

(c) Use this theorem to investigate the well-posedness of the variational formulation.

Solution.

(a) If the bilinear form $b(u,v)$ is continuous (trivially in our case), then the operator

$$B : U \to V', \quad \langle Bu, v\rangle := b(u,v), \quad v \in V,\, u \in U\,,$$

is always well-defined, linear and continuous. The map setting b into B is an isometric isomorphism. The conjugate operator,

$$B' : V'' \sim V \to U', \quad \langle B'v, u\rangle = b(u,v) \quad u \in U, v \in V\,,$$

is also well-defined, linear and continuous with the norm equal to that of B.

(b) If the bilinear form satisfies the inf-sup condition,

$$\sup_{v\in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U \quad \Leftrightarrow \quad \|Bu\|_{V'} \geq \gamma\|u\|_U$$

and $l \in V'$ vanishes on the null space of the transpose operator,

$$l(v) = 0 \quad \forall v \in V_0 := \{v \in V : b(w,u) = 0 \quad \forall w \in U\}\,,$$

then there exists a unique solution u to the variational problem and

$$\|u\|_U \leq \gamma^{-1}\|l\|_{V'}\,.$$

(c) We first prove the inf-sup condition. It is sufficient to find a $v \in H^1(0,1)$ such that $A^*v = u$ and

$$\|v\| \leq C\|A^*v\| = C\|u\|.$$

Once we control the L^2-norm of v, we control also the L^2-norm of its derivative,

$$\|v'\| \leq \|\underbrace{-v' + v}_{A^*v}\| + \|v\| \leq (1+C)\|A^*v\| = (1+C)\|u\|\,,$$

and, consequently,

$$\|v\|^2_{H^1(0,1)} = \|v\|^2 + \|v'\|^2 \leq \underbrace{\left((1+C)^2 + C^2\right)}_{C_1^2}\|u\|^2\,.$$

We have then

$$\sup_{v} \frac{|b(u,v)|}{\|v\|_{H^1}} \geq \frac{\|u\|^2_{L^2}}{\|v\|_{L^2}} \geq \frac{1}{C_1}\frac{\|u\|^2_{L^2}}{\|u\|_{L^2}} = \frac{1}{C_1}\|u\|_{L^2}\,.$$

Next, we determine the null space of the transpose operator. Clearly,

$$0 = \int_0^1 u A^* v \quad \forall u \in L^2(0,1) \quad \Rightarrow \quad A^* v = 0 \,.$$

This gives,

$$\mathcal{N}(B') = \{ce^x \, : \, c \in \mathbb{R}\} \,.$$

Consequently, by the Babuška-Nečas Theorem, for every $l \in (H^1(0,1))'$ that satisfies the compatibility condition

$$l(e^x) = 0 \,,$$

the variational problem has a unique solution u that depends continuously upon l. Note that the right-hand side may be more general than an L^2-function. For the L^2-function f,

$$l(v) = \int_0^1 f v \,,$$

so the function f must be L^2-orthogonal to e^x.

Finding a solution $v \in H^1(0,1)$, $A^* v = u \in L^2(0,1)$ is an undetermined problem. We may fix v by adding an extra BC: $v(0) = 0$. You can now find v explicitly (this is an elementary problem), or you can consider an auxiliary problem

$$\begin{cases} v \in H^1(0,1),\ v(0) = 0 \,, \\ Lv := -v' + v = u \,. \end{cases}$$

By the same argument as in the previous problem, operator L is bounded below,

$$\| - v' + v\|^2 = \|v'\|^2 + v(1)^2 + \|v\|^2 \geq \|v\|^2 \,.$$

The adjoint,

$$D(L^*) := \{u \in H^1(0,1) \, : \, u(1) = 0\}, \qquad L^* u = -u' + u \,,$$

has a trivial null space. The Closed Range Theorem for Closed Operators implies thus that there exists a unique solution $v \in D(L)$, $Lv = A^* v = u$, and $\|v\| \leq \|u\|$.

6. Nonlinear equations. Let X be a Banach space and $T : X \to X$ a bounded linear operator. Let $g : X \to X$ be a nonlinear mapping that is C^1 and has $g(0) = 0$ and $Dg(0) = 0$. For $f \in X$, we want to solve

$$F(u) = u + Tg(u) = f$$

We consider the map $G(u) = u + \alpha(F(u) - f)$ for some $\alpha \in \mathbb{R}$.

(a) Show that $G(u)$ is a contractive map for small enough u and properly chosen α.

(b) Use the Banach contraction mapping theorem to show that there is a solution to $F(u) = f$, provided f is sufficiently small.

(c) Compute $DF(u)(v)$ from the definition of the Fréchet derivative.

(d) Solve $F(u) = f$ using the inverse function theorem, provided f is sufficiently small.

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Solution.

(a) Let $u, v \in X$ and compute

$$G(u) - G(v) = u - v + \alpha(F(u) - F(v)) = (1+\alpha)(u-v) + \alpha T(g(u) - g(v))\,,$$

so that

$$\|G(u) - G(v)\| \le |1+\alpha|\,\|u-v\| + |\alpha|\,\|T\|\,\|g(u) - g(v)\|\,.$$

Since $Dg(0) = 0$ and g is C^1, given $\epsilon > 0$, there exists $\delta > 0$ such that for $w \in B_\delta(0)$, $\|Dg(w)\| \le \epsilon$. Therefore the mean value theorem shows that

$$\|g(u) - g(v)\| \le \epsilon\|u - v\| \quad \forall u, v \in B_\delta(0)\,.$$

Take, for example, $\alpha = -\frac{1}{2}$ and $\frac{1}{2}\epsilon\,\|T\| < \frac{1}{4}$ (which defines δ). Then G is contractive (with constant $\frac{3}{4}$) on $B_\delta(0)$.

(b) It remains to show that $G : B_\delta(0) \to B_\delta(0)$. Compute

$$\|G(u)\| \le \|G(u) - G(0)\| + \|G(0)\| \le \tfrac{3}{4}\|u\| + \|\alpha f\|\,.$$

Requiring $\|f\| < \dfrac{\delta}{4|\alpha|}$ completes the proof.

(c) We compute

$$\begin{aligned} F(u+v) - F(u) &= v + T(g(u+v) - g(u)) = v + T(Dg(u)(v) + R_g(u,v)) \\ &= v + T(Dg(u)(v)) + TR_g(u,v)\,, \end{aligned}$$

where $\|R_g(u,v)\| = o(\|v\|)$. But then $\|TR_g\| \le \|T\|\,\|R_g\| = o(\|v\|)$, so

$$DF(u)(v) = v + TDg(u)(v)\,.$$

(d) We note that F is C^1 and $DF(0) = I$ is invertible. Thus the inverse function theorem gives open sets $U, V \subset X$ such that $0 \in U$ and $F(0) = 0 \in V$ such that F is a diffeomorphism from U to V. Thus we can solve the problem for $f \in V$.

CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 28, 2021, about any 3 hours from 9:00 a.m. to 3:00 p.m.

You may use the class textbooks and your own notes on this exam.

Work any 5 of the following 6 problems.

1. Let the field be real and $\mathbb{P}$ denote the vector space of all polynomials in $x \in \mathbb{R}$; that is, $\mathbb{P} = \Big\{p(x) = \sum_{k=0}^{n} c_k x^k : n$ is a nonnegative integer and $c_k \in \mathbb{R}\Big\}$. Let $\|\cdot\| : \mathbb{P} \to [0,\infty)$ be defined for such p as $\|p\| = \max_{0\le k\le n} |c_k|$.

(a) Show $\|\cdot\|$ is a norm on $\mathbb{P}$.

(b) Show that the NLS $(\mathbb{P}, \|\cdot\|)$ is not complete.

(c) Let $m \ge 0$ and $T_m : \mathbb{P} \to \mathbb{R}$ be defined by $T_m p = \sum_{k=0}^{\min(m,n)} c_k$, which is clearly linear. Show that each T_m is bounded.

(d) Since $\mathbb{P}$ is not Banach, the Uniform Boundedness Principle need not hold. In fact, show that $\sup_m |T_m p| < \infty$ for each $p \in \mathbb{P}$ but $\sup_m \|T_m\| = \infty$.

2. Let Ω be some set and $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space of functions $f : \Omega \to \mathbb{F}$ ($\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). Suppose that there is a constant $C(x)$ such that

$$|f(x)| \le C(x)\|f\| \quad \text{for all } f \in H.$$

(a) Show that if $f, g \in H$ and $x \in \Omega$, then $|f(x) - g(x)| \le C(x)\|f - g\|$.

(b) Show that there exists a function $K : \Omega \times \Omega \to \mathbb{F}$ (called a *reproducing kernel*) such that for each fixed $x \in \Omega$, $K(\cdot, x) \in H$ and

$$f(x) = \langle f, K(\cdot, x)\rangle \quad \text{for all } f \in H.$$

[Hint: Use the Riesz representation theorem.]

(c) Show that $K(x, y) = \overline{K(y, x)}$ (i.e., K is conjugate symmetric). Be sure to justify that $K(x,\cdot) \in H$ for each $x \in \Omega$.

3. Let H be a complex Hilbert space and A a bounded linear operator on H. Define $|A| = (A^*A)^{1/2}$.

(a) Show that $|A|$ is a well defined, bounded linear, self-adjoint operator. [Hint: Use Theorem 4.26.]

(b) Show that $\|\,|A|x\,\| = \|Ax\|$ for all $x \in H$.

(c) Show that $H = \overline{R(|A|)} \oplus N(|A|)$ and that $N(|A|) = N(A)$.

4. Half Laplacian in $\mathbb{R}$. Let $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : \ y > 0\}$. For $u \in H^1(\mathbb{R}^2_+)$, we denote by $\bar{u}$ the Fourier transform in x only, i.e.,

$$\bar{u}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,y) e^{-ix\xi}\, dx.$$

Take $f \in H^1(\mathbb{R})$, and consider u the solution to

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & (x,y) \in \mathbb{R}^2_+, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

(a) Find the equation verified by $\bar{u}$.

(b) Show that there exists a unique solution of (1) such that $\nabla u \in L^2(\mathbb{R}^2_+)$, and give a formula for $\bar{u}$. [Hint: Solutions to the ODE $y'' - \omega^2 y = 0$ are of the form $Ae^{-\omega t} + Be^{\omega t}$.]

(c) For $f \in H^1(\mathbb{R})$, we define $\Delta^\alpha f$, for $0 < \alpha < 1$ a real number, through the Fourier transform as $\widehat{\Delta^\alpha f} = |\xi|^{2\alpha} \hat{f}$. Show that for u solving (1), we have

$$-\partial_y u(x,0) = \Delta^{1/2} f.$$

(d) Show that

$$\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx\, dy = \int_{\mathbb{R}} f \Delta^{1/2} f\, dx = \int_{\mathbb{R}} |\Delta^{1/4} f|^2\, dx.$$

5. Let $\Omega \in \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary, $f \in L^2(\Omega)$, and $\alpha > 0$. Consider the boundary value problem

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega. \end{cases}$$

(a) For this problem, formulate a variational principle

$$B(u,v) = (f,v) \qquad \forall v \in H^1(\Omega).$$

(b) Show that this problem has a unique weak solution.

6. Given $I = [0,b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that

$$\begin{cases} u'(s) = g(s) f(u(s)) & \text{for a.e. } s \in I, \\ u(0) = \alpha, \end{cases} \tag{2}$$

where $\alpha \in \mathbb{R}$ is a given constant, $g \in L^p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose that f is Lipschitz continuous and satisfies $f(0) = 0$.

(a) Consider the functional

$$F(u) = \alpha + \int_0^s g(\sigma) f(u(\sigma))\, d\sigma.$$

Show that F maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ is the solution to (2) if and only if it is a fixed point of F.

(b) Show that there exists b small enough, not depending on α, such that F has a unique fixed point in $C^0(I)$.

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1. $\mathbb{P} = \{ p = \sum_{k=0}^{n} c_k x^k \}$

$\|p\| = \max |c_k|$

(a) Norm (i) $\|p\| \geq 0$, $\|p\| = 0 \iff c_k = 0 \ \forall k$

$\iff p = 0$

(ii) $\|cp\| = \|\sum c c_k x^k\| = \max |c c_k|$

$= |c| \max |c_k| = |c| \, \|p\|$

(iii) $\|p+q\| = \max |c_k + d_k|$

$\leq \max |c_k| + \max |d_k| = \|p\| + \|q\|$

(b) Let $p_n = 1 + \frac{1}{2}x + \dots + \frac{1}{n}x^n$

Then $\{p_n\}$ is Cauchy: $(m > n)$

$\|p_n - p_m\| = \frac{1}{n+1} \longrightarrow 0$

But $p_n \not\to p$ with finite degree

(if $\deg p = m$, then $\|p_n - p\| \geq \frac{1}{m+1}$)

Thus $\mathbb{P}$ not complete

(c) $T_m p = \sum_{k=0}^{\min(n,m)} c_k$

$|T_m p| \leq \sum_{k=0}^{\min(n,m)} |c_k| \leq \min(n,m) \|p\|$

$\leq m \|p\|$

(d) $\sup_m |T_m p| \leq n \|p\| < \infty$

$\sup_m \|T_m\| \geq \sup_m \frac{|T_m p|}{\|p\|}$, $p = 1 + x + x^2 + \dots + x^n$

$\geq \sup_m |T_m p| = \min(n,m) = n \longrightarrow \infty$

2. $H = \{f : \Omega \to \mathbb{F}\}$

$|f(x)| \leq C(x)\,\|f\| \quad \forall f \in H$

(a) $f, g \in H,\ x \in \Omega \Rightarrow$

$|f(x) - g(x)| = |(f-g)(x)| \leq C(x)\,\|f-g\|$

(b) Let $T_x : H \to \mathbb{F}$ be $T_x f = f(x)$

Then T_x is a linear fcnal (by def'n of $+$, sc. mult. of fcns).

Riesz $\Rightarrow \exists\ g_x = K(\cdot, x) \in H$ s.t.

$f(x) = \langle f, K(\cdot, x)\rangle \quad \forall f \in H.$

(c) $K(\cdot, x) \in H \Rightarrow$ (by (b))

$$K(y,x) = \langle K(\cdot, x), K(\cdot, y)\rangle$$
$$= \overline{\langle K(\cdot, y), K(\cdot, x)\rangle} = \overline{K(x,y)}.$$

Note:

$K(x, \cdot) = \overline{K(\cdot, x)} \in H.$

3. H complex Hilbert. $A \in B(H,H)$. $|A| = (A^*A)^{1/2}$

(a) Let $T = A^*A \in B(H,H)$

$$\langle Tx, y\rangle = \langle A^*Ax, y\rangle = \langle Ax, Ay\rangle$$
$$= \langle x, A^*Ay\rangle = \langle x, Ty\rangle$$
$$\Rightarrow \quad T = T^*$$

$$\langle Tx, x\rangle = \|Ax\|^2 \geq 0 \Rightarrow T \geq 0$$

Thm 4.26 $\Rightarrow$ T has a unique pos. sq. root $(A^*A)^{1/2} \in B(H,H)$

Since $(A^*A)^{1/2} \geq 0$, it is self-adjoint.

(b) $\| |A|x \|^2 = \langle |A|x, |A|x\rangle$
$$= \langle |A|^2 x, x\rangle = \langle Tx, x\rangle$$
$$= \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2$$

(c) Let $R = \overline{R(|A|)}$

Then $H = R \oplus R^\perp$

$x \in R^\perp \Leftrightarrow \langle x, y\rangle = 0 \quad \forall y \in \overline{R(|A|)}$
$$\Leftrightarrow \langle x, y\rangle = 0 \quad \forall y \in R(|A|)$$
$$\Leftrightarrow \langle x, |A|z\rangle = 0 \quad \forall z \in H$$
$$\Leftrightarrow \langle |A|x, z\rangle = 0 \quad \forall z \in H$$
$$\Leftrightarrow \quad x \in N(|A|)$$

Thus $R^\perp = N(|A|)$ and
$$H = \overline{R(|A|)} \oplus N(|A|)$$

But
$$x \in N(|A|) \Leftrightarrow \| |A|x \| = 0 \Leftrightarrow \|Ax\| = 0$$
$$\Leftrightarrow \quad x \in N(A)$$

So $N(|A|) = N(A)$.

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4. $\mathbb{R}^2_+ = \{(x,y) : y>0\}$. $\bar{u} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x,y)\, e^{-ix\xi}\, dx$

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & \mathbb{R}^2_+ \\ u(x,0) = f(x) \in H^1 \end{cases}$$

(a)

$$\left.\begin{matrix} \underbrace{\overline{\partial_x^2 u}}_{= -|\xi|^2 \bar{u}} + \partial_y^2 \bar{u} = 0 \end{matrix}\right\} \Rightarrow \partial_y^2 \bar{u} - |\xi|^2 \bar{u} = 0$$

(b) Note: $\bar{u} = A e^{-|\xi|y} + B e^{+|\xi|y}$

But $\bar{u}(\xi,0) = \hat{f}$ & $\bar{u}$ blows up if $B \neq 0$

so $\bar{u}(\xi,y) = \hat{f}(\xi)\, e^{-|\xi|y}$

$\Rightarrow$

$$u(x,y) = \mathcal{F}_\xi^{-1}\left(\hat{f}(\xi)\, e^{-|\xi|y}\right)$$

$$= (2\pi)^{-1/2} f * \mathcal{F}_\xi^{-1}\left(e^{-|\xi|y}\right)$$

(c) $\widehat{\Delta^\alpha f} = |\xi|^{2\alpha} \hat{f}$

$$-\partial_y u(x,y) = -(2\pi)^{-1/2} f * \left(\partial_y \mathcal{F}_\xi^{-1}\left(e^{-|\xi|y}\right)\right)$$

$$(2\pi)^{-1/2}\left(f * \left(\partial_y \mathcal{F}_\xi^{-1}\left(e^{-|\xi|y}\right)\right)\right)^\wedge = \hat{f}\left(\partial_y \mathcal{F}_\xi^{-1}\left(e^{-|\xi|y}\right)\right)^\wedge$$

$$= -|\xi|\, \hat{f}\left(\mathcal{F}_\xi^{-1}\left(e^{-|\xi|y}\right)\right)^\wedge = -|\xi|\, \hat{f}\, e^{-|\xi|y}$$

as $y \to 0^+$, we have

$$-\partial_y u(x,0) = -\left(-|\xi|\, \hat{f}\right)^\vee = \Delta^{1/2} f$$

(d)

$$\int_{\mathbb{R}^2_+} |\nabla u|^2\, dx\,dy = (\nabla u, \nabla u)_{\mathbb{R}^2_+} = -(\Delta u, u)_{\mathbb{R}^2_+} + \int_{y=0} \nabla u \cdot \nu\, u$$

$$= \int_{\mathbb{R}} \Delta^{1/2} f\, f$$

$$= \int_{\mathbb{R}} \left(\Delta^{1/2} f\right)^\wedge \hat{f} = \int_{\mathbb{R}} |\xi|^{1/2} \hat{f}\, |\xi|^{1/2} \hat{f}$$

$$= \int_{\mathbb{R}} |\Delta^{1/4} f|^2\, dx$$

5. $\alpha > 0,\ f \in L^2$ $\begin{cases} -\Delta u + u = f & , \Omega \\ \partial_\nu u + \alpha u = 0 & , \partial\Omega \end{cases}$

(a) $\underbrace{(-\Delta u, v)} + (u, v) = (f, v)$

$$= (\nabla u, \nabla v) - \langle \nabla u \cdot \nu, v \rangle$$

$$= (\nabla u, \nabla v) + \langle \alpha u, v \rangle$$

$\Rightarrow$

$$B(u,v) = (\nabla u, \nabla v) + \alpha \langle u, v \rangle$$

$$= (f, v) \qquad \forall v \in H^1$$

where we want $u \in H^1$ as well.

(b) Use Lax-Milgram.

(f, v) gives a cont lin. fcnal for $f \in L^2 \subseteq (H^1)^*$

$$|B(u,v)| \leq \|\nabla u\| \|\nabla v\| + \alpha \|u\|_{H^1} \|v\|_{H^1}$$

$$\leq M \|u\|_{H^1} \|v\|_{H^1}$$

$$|B(u,u)| = \|\nabla u\|^2 + \alpha \|u\|^2_{L^2(\partial\Omega)} \overset{?}{\geq} \gamma \|u\|_{H^1}$$

We need a Poincaré ineq:

$$\|u\|_{L^2} \overset{?}{\leq} C(\|\nabla u\| + \|u\|_{L^2(\partial\Omega)})$$

Suppose not. Then $\exists\, u_n$ st

$$1 = \|u_n\|_{L^2} \geq n(\|\nabla u_n\| + \|u_n\|_{L^2(\partial\Omega)})$$

$\Rightarrow \quad u_n \rightharpoonup u \quad H^1 \quad (u_n \to u \quad L^2)$

$\nabla u_n \to 0 \quad L^2, \quad u_n|_{\partial\Omega} \to 0 \quad L^2(\partial\Omega)$

$\Rightarrow \quad u = 0$

But $\|u\| = 1$, contradiction.

6. $I=[0,b]$ $\quad \begin{cases} u'(s) = g(s) f(u(s)) & \text{, a.e. } s \in I \\ u(0) = \alpha \in \mathbb{R} \end{cases}$

$g \in L^p(I)$, $p \geq 1$; $f: \mathbb{R} \to \mathbb{R}$, $f(0)=0$, Lipschitz

(a)

$$F(u) = \alpha + \int_0^s g(\sigma) f(u(\sigma))\, d\sigma$$

$$u \in C^0 \Rightarrow F(u) \in C^0 \Rightarrow F(u) \in L^p$$

$$(F(u))' = g(s) \underbrace{f(u(s))}_{\in L^\infty \text{ since Lipschitz}} \in L^p$$

If

$$u = \alpha + \int_0^s g(\sigma) f(u(\sigma))\, d\sigma$$

Then $\begin{cases} u' = g(s) f(u(s)) \\ u(0) = \alpha \end{cases}$

(b) $$\|F(u) - F(v)\|_{L^\infty} = \left\| \int_0^s g(\sigma) (f(u) - f(v))\, d\sigma \right\|_{L^\infty}$$

$$\leq \|g\|_{L^p(0,b)} \|f(u) - f(v)\|_{L^{q}(0,b)}$$

$$\leq \underbrace{\|g\|_{L^p(0,b)} L}_{\theta} \|u - v\|_{L^\infty}$$

$\theta < 1$ if b small enough. (say $\theta = \frac{1}{2}$)

$\Rightarrow F$ contractive

$$\|F(u)\|_{L^\infty} = \|F(u) - F(0) + \alpha\|_{L^\infty}$$

$$\leq \theta \|u\|_{L^\infty} + \alpha \leq \theta R + \alpha \overset{?}{\leq} R$$

$\Rightarrow \alpha \leq (1-\theta) R \Rightarrow R = \frac{\alpha}{1-\theta} = 2\alpha$

Thus $F: \overline{B_R(0)} \to \overline{B_R(0)}$

and $\exists!$ fixed pt. in $\overline{B_R(0)}$

CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 31, 2022, 9:00 a.m. to 12:00 noon

Work on any 5 of the following 6 problems.

1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\int_\Omega dx = 1$. We consider a real base field and $X \in L^2(\Omega)$ as a *random variable* with *mean* $\mu(X) = \int_\Omega X(x)\,dx$ and *standard deviation* $\sigma(X) = \|X - \mu(X)\|_{L^2(\Omega)}$. The *covariance* of $X, Y \in L^2(\Omega)$ is $\operatorname{cov}(X,Y) = \langle X - \mu(X), Y - \mu(Y)\rangle_{L^2(\Omega)}$.

(a) State the domain and range of μ, σ, and cov. Why is $\mu \in (L^2(\Omega))^*$?

(b) Show that σ is a seminorm on $L^2(\Omega)$. Why is it *not* a norm?

(c) Show that $|\operatorname{cov}(X,Y)| \le \sigma(X)\,\sigma(Y)$.

(d) We denote the *probability* that $X \ge \alpha$ as $\operatorname{Prob}(X \ge \alpha) = \int_{\{x : X(x) \ge \alpha\}} dx$. Show Markov's inequality: $\operatorname{Prob}(X \ge \alpha) \le \frac{1}{\alpha}\mu(X)$.

2. Let H be a separable, infinite dimensional, complex Hilbert space and T a compact, self-adjoint operator on H. The Hilbert-Schmidt and spectral theorems tell us that there is a maximal orthonormal set of eigenvectors u_n with corresponding eigenvalues λ_n, $n = 1, 2, \ldots$. Let $P_n : H \to H$ be projection onto $\operatorname{span}\{u_n\}$.

(a) Show that for all $x \in H$, $P_n x = \langle x, u_n\rangle u_n$, $x = \sum_n P_n x$, and $T = \sum_n \lambda_n P_n$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the property that $f(\lambda) \to 0$ as $\lambda \to 0$. Define $f(T) : H \to H$ by

$$f(T) = \sum_n f(\lambda_n)\, P_n.$$

Show that $f(T)$ is well defined (i.e., the series converges). [Hint: Use Bessel's inequality.]

(c) Show that if $f(x) = x^2$, then $f(T) = T^2$.

3. Let $T : \mathcal{D}((-1,1)^2) \to \mathcal{D}(-1,1)$ be defined by $(T\varphi)(x) = \varphi(x,0)$.

(a) Show that T is a (sequentially) continuous linear operator.

(b) Note that the dual operator $T^* : \mathcal{D}'(-1,1) \to \mathcal{D}'((-1,1)^2)$. Determine $T^*(\delta_0)$ and $T^*(\delta_0')$, where δ_0 is the usual Dirac point distribution at 0 in one space dimension.

4. Let $\Omega \subset \mathbb{R}^2$ be an open, connected, and bounded domain with a smooth boundary containing 0. Let

$$X = \{f \in W^{1,3}(\Omega) \,:\, f(0) = 0\}.$$

(a) Use the Sobolev Embedding Theorem to conclude that $X \subset C^0(\Omega)$ and that $X \ne W^{1,3}(\Omega)$ is a Banach space.

(b) Prove the Poincaré-like inequality $\|f\|_{L^3(\Omega)} \le C\|\nabla f\|_{L^3(\Omega)}$, for some constant C independent of $f \in X$.

5. Let $f \in L^2(\mathbb{R}^d)$ and consider the problem

$$-\Delta u + u = f \quad \text{in } \mathbb{R}^d.$$

(a) Find the variational problem associated to the PDE.

(b) Use the Lax Milgram Theorem to show the existence and uniqueness of a solution in $H^1(\mathbb{R}^d)$ to the variational problem.

(c) Using the Fourier transform, show that the solution is actually in $H^2(\mathbb{R}^d)$.

6. Given $\alpha \in \mathbb{R}$, consider the problem of finding u such that

$$\begin{cases} u'(t) = \dfrac{u(t)}{1 + u^2(t)}, \\ u(0) = \alpha. \end{cases}$$

(a) By integrating, rewrite the differential equation in the fixed-point form $u = F(u)$ for an appropriate functional F.

(b) Show that F maps $C^0([0,T])$ into $C^0([0,T])$ for any $T > 0$.

(c) Show that the problem has a unique solution $u \in C^0([0,T])$ for sufficiently small but positive T.

1. $\int_\Omega dx = 1$, $X \in L^2(\Omega)$, $\mu(X) = \int X\,dx$, $\sigma(X) = \|X - \mu(X)\|$

$\mathrm{cov}(X,Y) = \langle X-\mu(X), Y-\mu(Y)\rangle$

(a) $\mu : L^2(\Omega) \to \mathbb{R}$, $\sigma : L^2(\Omega) \to \mathbb{R}$, $\mathrm{cov} : L^2(\Omega)\times L^2(\Omega) \to \mathbb{R}$

$\mu(\alpha X + Y) = \int_\Omega (\alpha X + Y) = \alpha \int X + \int Y = \alpha\mu(X) + \mu(Y)$

is linear, and $|\mu(X)| = \int X\,dx \le \|X\|\cdot\|1\| = \|X\|$

is bounded, so $\mu \in (L^2)^*$.

(b) $\sigma : L^2 \to [0,\infty)$

(i) $\sigma(\alpha X) = \|\alpha X - \mu(\alpha X)\| = \|\alpha(X - \mu(X))\|$

$= |\alpha|\,\|X - \mu(X)\| = |\alpha|\,\sigma(X) \quad \forall\, \alpha \in \mathbb{R}$

(ii) $\sigma(X+Y) = \|X+Y-\mu(X+Y)\| = \|X-\mu(X) + Y - \mu(Y)\|$

$\le \|X-\mu(X)\| + \|Y-\mu(Y)\| = \sigma(X) + \sigma(Y)$.

Note: if $X = \text{constant} \neq 0$, $\mu(X) = X$, so

$\sigma(X) = 0$ but $X \neq 0$.

(c) $|\mathrm{cov}(X,Y)| = |\langle X-\mu(X), Y-\mu(Y)\rangle|$

$\le \|X-\mu(X)\|\,\|Y-\mu(Y)\| = \sigma(X)\,\sigma(Y)$

(d) $\mathrm{Prob}(X \ge \alpha) = \int_{\{x : X(x) \ge \alpha\}} dx$

$\le \int_{\{x : X \ge \alpha\}} \frac{X(x)}{\alpha}\,dx \le \frac{1}{\alpha}\int X\,dx = \frac{1}{\alpha}\mu(X)$.

since $X \ge 0$

2. $T \in C(H,H)$, $T = T^*$, $\{u_n\}$, $\{\lambda_n\}$, P_n = proj. onto span$\{u_n\}$.

(a) $P_n : H \to$ span$\{u_n\}$, so for $x \in H$,

$P_n x = c u_n$. But

$$\langle P_n x, u_m \rangle = \langle c u_n, u_m \rangle = c\,\delta_{n,m}$$
$$= \langle x, u_m \rangle \implies \langle x, u_n \rangle = c$$
$$\implies P_n x = \langle x, u_n \rangle u_n$$

$x = \sum_m c_m u_m$ (max. ON set)

$$\implies P_n x = \sum_m c_m P_n u_m = \sum_m c_m \delta_{nm} u_n = c_n u_n$$
$$\implies x = \sum_n P_n x$$

$$Tx = T\left(\sum_n P_n x\right) = \sum_n T(P_n x) = \sum_n \langle x, u_n \rangle \lambda_n u_n$$
$$\implies T = \sum_n \lambda_n P_n$$

(b) $f(T) = \sum_n f(\lambda_n) P_n$

Since $\lambda_n \to 0$, $\forall \delta > 0$ $\exists N > 0$ s.t. $|\lambda_n| \le \delta$ $\forall n \ge N$.

Moreover, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(\lambda)| \le \varepsilon$ $\forall |\lambda| \le \delta$.

Thus the tail of the sequence

$$\left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n \right\|^2 = \sup_{\|x\|=1} \left\| \sum_{n=N+1}^{\infty} f(\lambda_n) P_n x \right\|^2$$
$$\le \sup_{\|x\|=1} \sum_{n=N+1}^{\infty} \varepsilon^2 \| P_n x \|^2 = \sup_{\|x\|=1} \sum_{n=N+1}^{\infty} \varepsilon^2 |\langle x, u_n \rangle|^2$$
$$\le \varepsilon^2 \text{ by Bessel's ineq.}$$

(c) $$T^2 x = T\left(\sum_n \lambda_n P_n x\right) = \sum_m \lambda_m P_m \left(\sum_n \lambda_n P_n x\right)$$
$$= \sum_{m,n} \lambda_m \lambda_n P_m P_n x = \sum_n \lambda_n^2 P_n x$$
$$\implies T^2 = f(T)$$

3. $T : \mathcal{D}((-1,1)^2) \to \mathcal{D}(-1,1)$, $T\varphi(x) = \varphi(x,y)$

(a) Let $\varphi_n \to \varphi$ in $\mathcal{D}((-1,1)^2)$. Then

$$\| x^{\alpha_1} y^{\alpha_2} D_x^{\beta_1} D_y^{\beta_2}(\varphi_n - \varphi)\|_{L^\infty} \to 0 \quad \forall \alpha_1, \alpha_2, \beta_1, \beta_2$$

Now $\forall \alpha, \beta$

$$\| x^\alpha D^\beta (T(\varphi_n) - T(\varphi))\|_{L^\infty} = \| x^\alpha D^\beta (\varphi_n(\cdot,0) - \varphi(\cdot,0))\|_{L^\infty}$$
$$\leq \| x^\alpha D_x^\beta \varphi_n(\cdot,\cdot) - \varphi(\cdot,\cdot)\|_{L^\infty((-1,1)^2)} \to 0.$$

$\Rightarrow$ T is seq. cont. Linearity is clear.

(b) For $\mu \in \mathcal{D}'(-1,1)$, $\varphi \in \mathcal{D}((-1,1)^2)$

$$(T^*\mu)(\varphi) = \mu(T\varphi)$$

Thus

$$(T^*\delta)(\varphi) = \delta(T\varphi) = \varphi(0,0)$$

That is, $T^*\delta$ = Dirac dist. at $(0,0)$.

Moreover

$$(T^*\delta')(\varphi) = \delta'(T\varphi) = \delta'(\varphi(x,0))$$
$$= -\frac{\partial \varphi}{\partial x}(0,0)$$

4. Ω open, connected, bnded, $\ni 0$. $X = \{W^{1,3} : f(0) = 0\}$.

(a) $W_0^{1,3}(\Omega) \hookrightarrow C^0(\Omega)$

since $mp \leq d$, i.e. $3 \cdot 1 \geq 2$.

But $\partial\Omega$ smooth, so $W^{1,3}(\Omega) \hookrightarrow C^0(\Omega)$

Thus $X \subseteq C^0(\Omega)$ and $T_0 f = f(0)$

is well-defined cont. op. Thus

X closed under $+$, sc. add $\Rightarrow$ X Banach, $\neq W^{1,3}$

(b) Suppose $\|f_n\|_{L^3} = 1$ but $\|\nabla f_n\|_{L^3} \leq \frac{1}{n}$.

Then

$\|f_n\|_{W^{1,3}} \leq C \Rightarrow f_{n_i} \rightharpoonup f$ in $W^{1,3}$.

for a subsequence

Moreover $f_{n_i} \longrightarrow f$ in L^3

$\nabla f_{n_i} \longrightarrow 0$ in L^3

$\Rightarrow$ $f_{n_i} \longrightarrow f$ in $W^{1,3}$

But $\nabla f_{n_i} \to 0 \Rightarrow f_{n_i} \longrightarrow$ constant

Since $f_{n_i}(0) = 0$, $f_{n_i} \longrightarrow 0$

This contradicts that $\|f_{n_i}\|_{L^3} = 1$, so

$\|f\|_{L^3} \leq C \|\nabla f\|_{L^3}$ for some $C > 0$.

5. $-\Delta u + u = f \in L^2(\mathbb{R}^d)$

(a) Let $v \in \mathcal{D}(\mathbb{R}^d) \underset{\text{dense}}{\subseteq} H^1(\Omega)$. Then

$$(-\Delta u, v) = (\nabla u, \nabla v) \implies$$

$$\begin{cases} \text{Find } u \in H^1(\mathbb{R}^d) \text{ s.t.} \\ (\nabla u, \nabla v) + (u, v) = (f, v) \quad \forall v \in H^1(\mathbb{R}^d) \end{cases}$$

(b) $(f, v) = F(v)$, $F \in (H^1)^*$ since

$$|(f, v)| \le \|f\|_{L^2} \|v\|_{H^1}$$

$$|(\nabla u, \nabla v) + (u, v)| \le \|\nabla u\| \|\nabla v\| + \|u\| \|v\|$$
$$\le 2 \|u\|_{H^1} \|v\|_{H^1} \quad \text{continuous.}$$

$$(\nabla u, \nabla u) + (u, u) = \|u\|_{H^1}^2 \quad \text{coercive}$$

$\Rightarrow$ $\exists!$ sol'n in $H^1(\mathbb{R}^d)$.

(c) FT: $|\xi|^2 \hat{u} + \hat{u} = \hat{f}$

$$\Rightarrow \quad \hat{u} \in \frac{\hat{f}}{1 + |\xi|^2}$$

Now $\|u\|_{H^2} = \int (1 + |\xi|^2) |\hat{u}(\xi)|^2 \, d\xi$

$$= \int |\hat{f}(\xi)|^2 \, d\xi = \int |f(x)|^2 \, dx < \infty$$

$\Rightarrow$

$u \in H^2(\mathbb{R}^d)$

6. $u'(t) = \dfrac{u(t)}{1+u^2(t)}$, $u(0) = \alpha$

(a)

$$u(t) = \alpha + \int_0^t \frac{u(s)}{1+u^2(s)}\, ds \equiv F(u)$$

(b) $F : C^0([0,T]) \longrightarrow C^0([0,T])$

since the integral of a cont. fcn. is cont.

(c) We show F is a contraction on $X = \{u \in C^0([0,T]) : \|u\|_{L^\infty} \leq R\}$ for some $T, R > 0$.

$$\|F(u) - F(v)\|_{L^\infty} = \left\| \int_0^t \left(\frac{u}{1+u^2} - \frac{v}{1+v^2} \right) dt \right\|_{L^\infty}$$

$$= \left\| \int_0^t \frac{u + uv^2 - v - u^2 v}{(1+u^2)(1+v^2)}\, dt \right\|_{L^\infty} = \left\| \int_0^t \frac{1-uv}{(1+u^2)(1+v^2)} (u-v)\, dt \right\|_{L^\infty}$$

$$\leq T(2R+1) \|u-v\|_{L^\infty} \equiv \theta \|u-v\|_{L^\infty}$$

Moreover

$$\|F(u)\| = \|F(u) - F(0)\| + |\alpha| \leq T(2R+1)\|u\| + |\alpha|$$

$$\leq T(2R+1)R + |\alpha|$$

Want

$$\theta = T(2R+1) < 1, \quad T(2R+1)R + |\alpha| \leq R$$

Take $R = 2|\alpha| + 1 > 0$. Then

$$T < \frac{1}{4|\alpha|+3} \quad \text{and} \quad T \leq \frac{|\alpha|+1}{(4|\alpha|+3)(2|\alpha|+1)}$$

so take the minimum of these 2.

Note $T > 0$.

EXAM #____

CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 15, 2023, 9:00 a.m. to 12:00 noon

Work on any 5 of the following 6 problems.

1. Let X be a normed linear space and $M \subset X$ a linear subspace.

(a) State the Hahn-Banach Theorem for normed linear spaces.

(b) If M is closed and $x_0 \in X \setminus M$, use the Hahn-Banach Theorem to prove that there is some $f \in X^*$ satisfying $f(x_0) \neq 0$ and $f(x) = 0$ for any $x \in M$.

(c) If M is not necessarily closed, prove that for any $x_0 \in X$, $x_0 \in \overline{M}$ if and only if there is no bounded linear functional f on X satisfying $f(x) = 0$ for any $x \in M$ but $f(x_0) \neq 0$.

2. Open Mapping Theorem.

(a) State the Open Mapping Theorem.

(b) Suppose that $\|\cdot\|$ and $\|\cdot\|'$ are two norms on a vector space X. Suppose that both $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are complete and there is a constant $C > 0$ such that

$$\|x\| \leq C\|x\|' \quad \text{for all } x \in X.$$

From the Open Mapping Theorem, show that the two norms are equivalent.

(c) Use (b) to show that when $X = L^\infty([0,1])$, $(X, \|\cdot\|_{L^1})$ is *not* complete.

3. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$.

(a) For $\epsilon > 0$, let $\varphi_\epsilon(\mathbf{x}) = \epsilon^{-d}\varphi(\epsilon^{-1}\mathbf{x})$. Show that for $f \in C^0(\mathbb{R}^d)$,

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \varphi_\epsilon(\mathbf{x})\, f(\mathbf{x})\, d\mathbf{x} = C f(0)$$

for some constant C. Find the constant C.

(b) Show that for any $u \in \mathcal{D}'(\mathbb{R}^d)$ and any multi-index α, $D^\alpha u * \varphi = u * D^\alpha \varphi$.

4. Let Ω be a bounded domain with a smooth boundary and let ν be the unit normal vector on its boundary. Consider the solution (u, v) of the differential problem

$$\begin{aligned} u + \Delta^2 u + w &= f \quad \text{in } \Omega, \\ -\Delta w - u &= g \quad \text{in } \Omega, \\ u = \nabla u \cdot \nu &= 0 \quad \text{on } \partial\Omega, \\ w &= \gamma \quad \text{on } \partial\Omega. \end{aligned}$$

(a) Provide an appropriate weak form for the problem. In what Sobolev spaces should u, w, f, g, γ, and the test functions lie?

(b) Prove that there exists a unique solution to the problem.

5. Let $\phi(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $K(x) \in L^1(\mathbb{R})$. Use the contraction mapping principle to prove that the initial-value problem

$$\begin{aligned} \partial_t u &= K * u^2, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \phi(x) \end{aligned}$$

has a continuous and bounded solution $u = u(x,t)$, at least up to some time $T < \infty$.

6. For any $a \in \mathbb{R}$ and $b \in \mathbb{R}$, let the Rectified Linear Unit (ReLU) function $R_{a,b} : \mathbb{R} \to \mathbb{R}$ be

$$R_{a,b}(x) = \max(ax + b, 0).$$

Define

$$G = \left\{ \sum_{j=1}^{m} \alpha_j R_{a_j,b_j} \ : \ m \in \mathbb{N},\ \alpha_j, a_j, b_j \in \mathbb{R} \right\}.$$

Clearly G consists of piecewise linear functions. In fact, $\varphi \in G$, where

$$\varphi(x) = R_{0,1}(x) - R_{1,0}(x) + R_{1,-1}(x) - R_{-1,0}(x) + R_{-1,-1}(x) = \begin{cases} 0, & |x| \geq 1, \\ 1 - |x| & |x| \leq 1. \end{cases}$$

(a) Show that G is invariant to scaling ($x \mapsto \alpha x$) and translation ($x \mapsto x + c$).

(b) Show that if $g \in C([0,1])$, then

$$\int_0^1 R_{a,b}(x)\, g(x)\, dx = 0 \quad \forall\, a, b \in \mathbb{R} \qquad \Longrightarrow \qquad g = 0.$$

(c) Let S be the set of functions in G restricted to $[0,1]$. Show that S is dense in $L^2(0,1)$. [Hint: use the density of $C([0,1])$ in $L^2(0,1)$ and (b) to show that $S^\perp = \{0\}$.]

CSEM Area A-CAM Preliminary Exam (CSE 386C–D)

May 20, 2024, 9:00 a.m. to 12:00 noon

Work on any 5 of the following 6 problems.

1. Let X be an NLS and $Y \neq X$ a closed subspace.

(a) Define what we mean by the distance from $x \in X$ to Y, i.e., $\text{dist}(x, Y)$.

(b) If θ is given with $0 < \theta < 1$, prove that there is some $x \in X$ such that $\|x\| = 1$ and $\text{dist}(x, Y) \geq \theta$.

(c) Must there be a unique point $x \in X$ such that $\text{dist}(x, Y) = 1$? Why or why not?

2. The Riesz Representation Theorem states that if H is a Hilbert space and $L \in H^*$, then there is a unique $y \in H$ such that $Lx = \langle x, y \rangle$ for all $x \in H$.

(a) For a given $L \in H^*$, prove that the associated $y \in H$ is unique.

(b) For a given $L \in H^*$, $L \neq 0$, prove that the associated $y \in H$ exists. [Hint: Recall that if N is the null space of L, then we expect that $y \in N^\perp$. Let $z \in N^\perp$ and consider $u = (Lx)z - (Lz)x$.]

(c) Show that $\|L\|_{H^*} = \|y\|_H$.

3. Let H be a separable Hilbert space and T a positive operator on H. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal base for H and define the trace of T to be

$$\text{tr}(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle$$

and suppose this number is finite for T.

(a) Show that if S a positive operator on H such that $0 \leq T \leq S$, then $\text{tr}(T) \leq \text{tr}(S)$.

(b) Show the the trace of T is independent of which base is chosen. [Hint: Care must be taken when interchanging infinite sums, unless all the terms are positive. Use the operator $T^{1/2}$ to resolve this issue.]

(c) If T is also compact, show that $\text{tr}(T) = \sum_{n=1}^\infty \lambda_n$, where λ_n are the eigenvalues of T.

4. Let $f, g \in L^1(\mathbb{R}^d)$. Recall that a continuous function ϕ on $\mathbb{R}^d$ is said to vanish at infinity if for any $\epsilon > 0$, there is a compact set K_ϵ such that $|\phi(x)| < \epsilon$ for $x \notin K_\epsilon$. The subspace of all such continuous functions is denoted $C_v(\mathbb{R}^d)$.

(a) Prove that $C_v(\mathbb{R}^d)$ is a closed linear subspace of $L^\infty(\mathbb{R}^d)$.

(b) Prove that the Fourier transform $\hat{f} \in L^\infty(\mathbb{R}^d)$.

(c) Prove that $f * g \in L^1(\mathbb{R}^d)$ directly (i.e., do not use Young's inequality) and $\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$.

(d) Show that the Fourier transform $\widehat{f * g} = (2\pi)^{d/2} \hat{f} \hat{g}$.

5. Let $\Omega = [0,1]^d$, define

$$H^1_{\#}(\Omega) = \left\{ v \in H^1_{\text{loc}}(\mathbb{R}^d) : v \text{ is periodic of period 1 in each direction and } \int_\Omega v\,dx = 0 \right\}.$$

(a) Define precisely what it means for $v \in H^1(\mathbb{R}^d)$ to be periodic of period 1 in each direction.

(b) Define the natural inner product and norm that one should use on this space.

(c) Show that $H^1_{\#}(\Omega)$ is a Hilbert space.

6. Let $\Omega \in \mathbb{R}^d$ have a smooth boundary, V_n be the set of polynomials of degree up to n, for $n = 1, 2, \ldots$, and $f \in L^2(\Omega)$. Consider the problem: Find $u_n \in V_n$ such that

$$B(u_n, v_n) = (\nabla u_n, \nabla v_n)_{L^2(\Omega)} + (u_n, v_n)_{L^2(\Omega)} = (f, v_n)_{L^2(\Omega)} \qquad \text{for all } v_n \in V_n.$$

(a) Show that there exists a unique solution for any n, and that

$$\|u_n\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

(b) Show that there is $u \in H^1(\Omega)$ such that, for a subsequence, $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. Find a variational problem satisfied by u. Justify your answer.

(c) Show that $\|u - u_n\|_{H^1(\Omega)}$ decreases monotonically to 0 as $n \to \infty$.

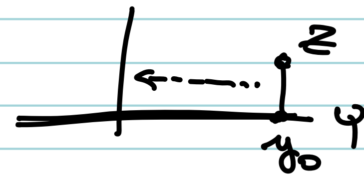
1. X NLS, $Y \subsetneq X$ closed.

(a) $\operatorname{dist}(x, Y) = \inf_{y \in Y} \|x - y\|$

(b) $0 < \theta < 1$.

Let $z \in X \setminus Y$.

Find $y_0 \in Y$ s.t.



$$\|z - y_0\| \le \frac{1}{\theta} \operatorname{dist}(z, Y) \qquad \left(\frac{1}{\theta} > 1\right)$$

Let $x = \dfrac{z - y_0}{\|z - y_0\|}$ have norm 1.

Now

$$\operatorname{dist}(x, Y) = \inf_{y \in Y} \left\| \frac{z - y_0}{\|z - y_0\|} - y \right\|$$

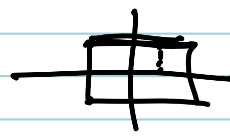
$$= \frac{1}{\|z - y_0\|} \inf_{y} \| z \underbrace{- y_0 - \|z - y_0\| y}_{\in Y} \|$$

$$= \frac{1}{\|z - y_0\|} \operatorname{dist}(z, Y)$$

$$\ge \frac{\theta}{\operatorname{dist}(z, Y)} \operatorname{dist}(z, Y) = \theta$$

(c) No. ① X NLS so not complete.

② In $\ell^\infty(\mathbb{R}^2)$ unit ball is a square, so not unique



or ③ In $\mathbb{R}^2$, get points "above" and "below",

2. Riesz. $Lx = \langle x, y \rangle$

(a) Suppose $Lx = \langle x, y \rangle = \langle x, z \rangle \quad \forall\, x \in H$

Then $\langle x, y - z \rangle = 0 \quad \forall x \in H$

$\Rightarrow \quad y - z = 0 \quad \Rightarrow \quad y = z$, thus unique.

(b) $L \neq 0$, $N = \mathrm{Null}(L)$.

Let $z \in N^{\perp}$ and $u = (Lx)z - (Lz)x$

Note: $Lu = (Lx)Lz - (Lz)Lx = 0 \Rightarrow u \in N$

Thus

$0 = \langle u, z \rangle = \langle (Lx)z, z \rangle - \langle (Lz)x, z \rangle$

$\Rightarrow \quad (Lx)\|z\|^2 = \langle (Lz)x, z \rangle = \langle x, \overline{Lz}\, z \rangle$

Let $y = \dfrac{\overline{Lz}\, z}{\|z\|^2}$, so $Lx = \langle x, y \rangle$.

(c) $\|L\| = \sup\limits_{\substack{x \in H \\ x \neq 0}} \dfrac{\|Lx\|}{\|x\|} = \sup\limits_{x} \dfrac{\langle x, y \rangle}{\|x\|} \leq \|y\|$

But $L\left(\frac{y}{\|y\|}\right) = \dfrac{\langle y, y \rangle}{\|y\|} = \|y\|$

$\leq \sup\limits_{x \in H} L\left(\frac{x}{\|x\|}\right) = \|L\|$

Thus $\|L\| = \|y\|$.

3. $T \geq 0$, $\quad tr(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle < \infty$

(a) If $0 \leq T \leq S$, then

$$\langle Te_n, e_n \rangle \leq \langle S e_n, e_n \rangle$$

sum to see $tr(T) \leq tr(S)$.

(b) Let $\{f_m\}_{m=1}^{\infty}$ be another ON base

$$tr_e(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle = \sum_{n=1}^{\infty} \|T^{1/2} e_n\|^2$$

$$= \sum_{n=1}^{\infty} \left\| T^{1/2} \left(\sum_{m=1}^{\infty} \langle e_n, f_m \rangle f_m \right) \right\|^2$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle e_n, f_m \rangle|^2 \, \| T^{1/2} f_m \|^2$$

$$= \sum_{m=1}^{\infty} \underbrace{\sum_{n=1}^{\infty} |\langle e_n, f_m \rangle|^2}_{\|f_m\|^2 = 1} \, \|T^{1/2} f_m\|^2$$

$$= \sum_{m=1}^{\infty} \|T^{1/2} f_m\|^2 = tr_f(T).$$

(c) $T \geq 0 \Rightarrow T$ self-adj. now compact $\Rightarrow$ $\exists$ ON base $\{e_n\}$ and e-values $\{\lambda_n\}$ s.t.

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle e_n$$

$$\Rightarrow \langle Te_n, e_n \rangle = \lambda_n \langle e_n, e_n \rangle = \lambda_n$$

$$\Rightarrow tr(T) = \sum_{n=1}^{\infty} \lambda_n$$

4. $f, g \in L^1(\mathbb{R}^d)$, $C_v = \{\phi \text{ cont.} : \forall \varepsilon, |\phi(x)| < \varepsilon \text{ for } x \notin K_\varepsilon\}$

(a) $\phi, \psi \in C_v$, $\lambda \in \mathbb{F}\ (\lambda \neq 0) \Rightarrow$

$$|\phi(x) + \psi(x)| < \varepsilon \quad \text{for} \quad x \notin K_\varepsilon^\phi \cup K_\varepsilon^\psi = K_\varepsilon \text{ compact}$$

$$|\lambda \phi(x)| < \varepsilon \quad \text{for} \quad x \notin K^\phi_{\varepsilon/\lambda}$$

$\Rightarrow \phi + \psi, \lambda\phi$ vanishes at ∞, so subspace

If $\phi_n \to \phi$ in L^∞, ϕ is continuous

If $\varepsilon > 0$ is given, choose $N \geq 0$ st.

$$\|\phi_n - \phi\|_{L^\infty} \leq \varepsilon \quad \forall n \geq N$$

Then

$$|\phi(x)| \leq |\phi_N(x) - \phi(x)| + |\phi_N(x)|$$

$$\leq \varepsilon + \varepsilon \quad \text{for} \quad x \notin K^{\phi_N}$$

$\Rightarrow \phi$ vanishes at ∞.

(b)

$$|\hat{f}(\xi)| = \frac{1}{(2\pi)^{d/2}} \left| \int f(x) e^{-ix\cdot\xi} dx \right| \leq \frac{1}{(2\pi)^{d/2}} \int |f(x)| dx$$

$$\leq \frac{1}{(2\pi)^{d/2}} \|f\|_{L^1} \to \hat{f} \in L^\infty$$

(c) $f * g(x) = \int f(x-y)\, g(y)\, dy$

$$\int |f*g(x)|\, dx \leq \int\int |f(x-y)|\, |g(y)|\, dy\, dx$$

$$= \int\int |f(x-y)|\, dx\, g(y)\, dy$$

$$= \int \|f\|_{L^1}\, g(y)\, dy = \|f\|_{L^1} \|g\|_{L^1}$$

(d)

$$\widehat{f*g}(\xi) = \frac{1}{(2\pi)^{d/2}} \int \int f(x-y) g(y)\, dy\, e^{-ix\xi} dx$$

$$= \frac{1}{(2\pi)^{d/2}} \int\int f(x-y)\, e^{-i(x-y)\xi} dx\; g(y) e^{-iy\xi} dy$$

$$= \int \hat{f}(\xi)\, g(y)\, e^{-iy\xi} dy$$

$$= (2\pi)^{d/2} \hat{f}\, \hat{g}$$

5. $\Omega = [0,1]^d$, $H^1_\#(\Omega) = \{ v \in H^1_{loc} : v \text{ per.}, \int_\Omega v\,dx = 0 \}$

(a) For almost every $x \in \Omega$ and for every $\eta \in \mathbb{Z}^d$,

$$f(x+\eta) = f(x)$$

(b) $$\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)}\,dx + \int_\Omega \nabla f \cdot \overline{\nabla g}\,dx$$

$$\|f\|^2_{H^1_\#} = \int_\Omega |f|^2\,dx + \int_\Omega |\nabla f|^2\,dx = \|f\|^2_{H^1(\Omega)}$$

(c) $H^1_\#$ is clearly a v.s.

If $\{f_n\} \in H^1_\#$ is Cauchy, then $\{f_n|_\Omega\}$ is Cauchy in $H^1(\Omega)$, so $\exists\, f \in H^1(\Omega)$ s.t.

$$\|f_n - f\|_{H^1(\Omega)} \to 0.$$

Moreover,

$$0 = \int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$$

Let $f(x+\eta) = f(x) \quad \forall\ \eta \in \mathbb{Z}^d, x \in \Omega$

Then $f \in H^1_\#$, and

$$\|f_n - f\|_{H^1_\#} \to 0.$$

Thus $H^1_\#$ is complete.

6. V_n polyn. deg. n

Find $u_n \in V_n$ s.t.

$$B(u_n,v_n) = (\nabla u_n, \nabla v_n) + (u_n, v_n) = (f, v_n) \quad \forall v_n \in V_n$$

(a) Lax-Milgram implies $\exists!$ sol'n, since the form $B(\cdot,\cdot)$ is cont. & coercive on H^1, $V_n \subseteq H^1$

Moreover, $v_n = u_n \Rightarrow$

$$\|u_n\|_{H^1}^2 = (f, u_n) \le \tfrac{1}{2}\|f\|^2 + \tfrac{1}{2}\|u_n\|^2$$

$$\Rightarrow \quad \|u_n\|_{H^1} \le \|f\|$$

(b) Banach Alaoglu $\Rightarrow$ $\exists$ subseq. s.t. $u_{n_k} \rightharpoonup u$ in H^1

Then $(\nabla u_{n_k}, \nabla v_n) + (u_{n_k}, v_{n_k}) = (f, v_n)$

$$\downarrow \qquad\qquad \downarrow$$

$$(\nabla u, \nabla v_n) + (u, v_n) = (f, v_n)$$

But polyn. dense in H^1, so

$$B(u,v) = (\nabla u, \nabla v) + (u,v) = (f, v) \quad \forall v \in H^1$$

(c) Now $V_n \supseteq V_{n-1}$, so

$$\|u-u_n\|_{H^1}^2 = B(u-u_n, u-u_n) \overset{\text{Galerkin }\perp}{=} B(u-u_n, u-u_{n-1})$$

$$\le \|u-u_n\|_{H^1} \|u-u_{n-1}\|_{H^1}$$

$\Rightarrow$

$$\|u-u_n\|_{H^1} \le \|u-u_{n-1}\|_{H^1}$$

Moreover, $V_n \subseteq H^1$ dense $\Rightarrow$

$$\|u-u_n\|_{H^1} \to 0.$$